

# DECOMPOSING HIGHLY EDGE-CONNECTED GRAPHS INTO PATHS OF ANY GIVEN LENGTH

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**ABSTRACT.** In 2006, Barát and Thomassen posed the following conjecture: for each tree  $T$ , there exists a natural number  $k_T$  such that, if  $G$  is a  $k_T$ -edge-connected graph and  $|E(G)|$  is divisible by  $|E(T)|$ , then  $G$  admits a decomposition into copies of  $T$ . This conjecture was verified for stars, some bistars, paths of length 3, 5, and  $2^r$  for every positive integer  $r$ . We prove that this conjecture holds for paths of any fixed length.

## 1. INTRODUCTION

A decomposition  $\mathcal{D} = \{H_1, \dots, H_k\}$  of a graph  $G$  is a set of pairwise edge-disjoint subgraphs of  $G$  that cover the edge set of  $G$ . If  $H_i$  is isomorphic to a fixed graph  $H$  for  $1 \leq i \leq k$ , then we say that  $\mathcal{D}$  is an  $H$ -decomposition of  $G$ . It is known that, when  $H$  is connected and contains at least 3 edges, the problem of deciding whether a graph admits an  $H$ -decomposition is NP-complete [12]. When  $H$  is a tree, Barát and Thomassen [3] proposed the following conjecture, that is the subject of our interest in this paper.

**Conjecture 1.1.** *For each tree  $T$ , there exists a natural number  $k_T$  such that, if  $G$  is a  $k_T$ -edge-connected graph and  $|E(G)|$  is divisible by  $|E(T)|$ , then  $G$  admits a  $T$ -decomposition.*

The following version of Conjecture 1.1 for bipartite graphs was shown by Barát and Gerbner [2], and independently by Thomassen [24], to be equivalent to Conjecture 1.1.

**Conjecture 1.2.** *For each tree  $T$ , there exists a natural number  $k'_T$  such that, if  $G$  is a  $k'_T$ -edge-connected bipartite graph and  $|E(G)|$  is divisible by  $|E(T)|$ , then  $G$  admits a  $T$ -decomposition.*

Most of the known results on Conjecture 1.1 were obtained by Thomassen [23, 21, 22, 24, 25]: it holds for stars, paths of length 3, a family of bistars, and for paths whose length is a power of 2. In 2014, we [8] proved that it holds for paths of length 5, and

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recently Merker [17] proved that it holds for all trees with diameter at most 4, and also for some trees with diameter at most 5, including paths of length 5.

In this paper we verify Conjecture 1.2 (and Conjecture 1.1) for paths of any given length. More specifically, we prove that, for  $P_\ell$ , the path of length  $\ell$ , we have  $k'_{P_\ell} \leq 4\ell^2 + 10\ell - 4$ , if  $\ell$  is odd; and  $k'_{P_\ell} \leq 26\ell + 8r - 8$ , with  $r = \max\{32(\ell - 1), \ell(\ell + 2)\}$ , if  $\ell$  is even.

In our proof (for  $P_\ell$ ) we use a generalization of a technique used by Thomassen [21] to obtain an initial decomposition into trails of length  $\ell$ . We also borrow some ideas from a technique that we used in [10] for regular graphs. A central part of this work concerns the “disentangling” of the undesired trails of our initial decomposition to construct a path decomposition.

The paper is organized as follows. In Section 2 we give some definitions, establish the notation and state some auxiliary results needed in the proof of our main results. In Section 3 we present our main tool, called *Disentangling Lemma*, that allows us to switch edges between the elements of a (special) trail decomposition so as to obtain a decomposition into paths. In Section 4 we prove that highly edge-connected graphs admit well-structured decompositions with good properties that we can explore in the rest of the proof. In Sections 5 and 6 we present the results used in Section 7 to obtain the decompositions into paths of fixed odd and even length, respectively. In Figure 1 we present a diagram that shows how the results (indicated in a rectangular box) are connected with each other, leading to the proof of our two main results, Theorems 7.1 and 7.4. In this diagram, an arrow from a box A to a box B indicates that the result in A is used to prove the result in B.

An extended abstract [9] of this work was accepted to EUROCOMB 2015. We have modified some previous terminology, but the techniques and results are essentially those we have mentioned in the extended abstract. This work grew out from our previous work on decomposition into paths of length five [6]. The reader may find useful to see the simpler ideas presented in this previous work, to get a better understanding of the technique used in this paper.

## 2. NOTATION AND AUXILIARY RESULTS

The basic terminology and notation used in this paper are standard (see, e.g. [5, 11]). All graphs considered here are finite and have no loops. Let  $G = (V, E)$  be a graph. A *path*  $P$  in  $G$  is a sequence of distinct vertices  $P = v_0v_1 \cdots v_\ell$  such that  $v_iv_{i+1} \in E$ , for  $0 \leq i \leq \ell - 1$ . The *length* of a path  $P$  is the number of its edges. The path of length  $\ell$ , also called  $\ell$ -*path*, is denoted by  $P_\ell$ . It is also convenient to refer to a path  $P = v_0v_1 \cdots v_\ell$  as the subgraph of  $G$  induced by the edges  $v_iv_{i+1}$  for  $i = 0, \dots, \ell - 1$ .

We denote by  $d_G(v)$  the *degree* of a vertex  $v \in V$  and, when  $G$  is clear from the context, we write  $d(v)$ . Given  $F \subset E$ , we denote by  $G[F]$  the subgraph of  $G$  induced by  $F$ , and we also denote by  $d_F(v)$  the number of edges in  $F$  that are incident to  $v$ . An *orientation*  $O$

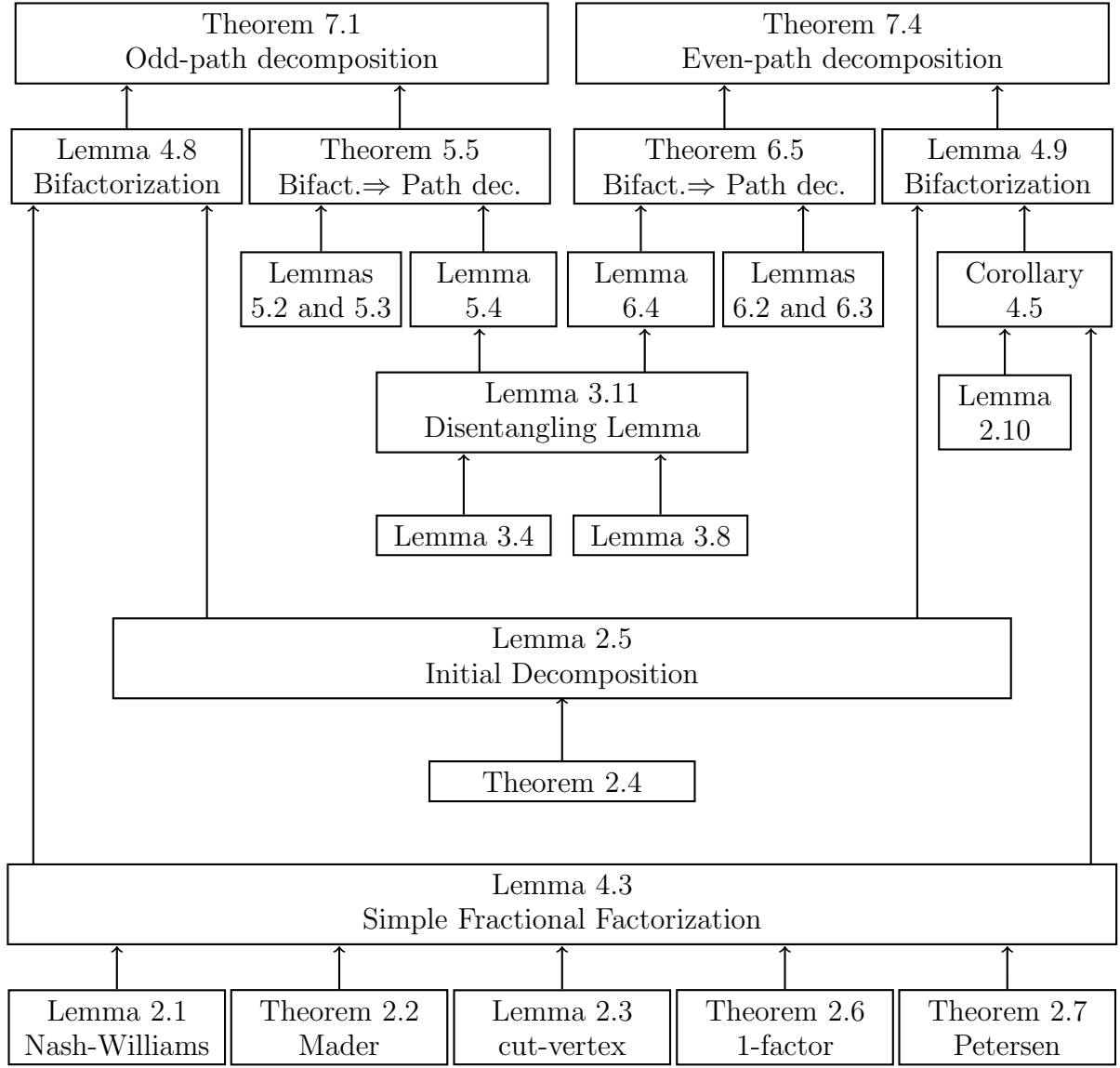


FIGURE 1. A diagram showing how the auxiliary results are used to build up the proof of the main results.

of a subset  $F \subset E$ , is an assignment of a direction (from one of its vertices to the other) to each edge in  $F$ . If an edge  $e = uv$  in  $F$  is directed from  $u$  to  $v$ , we say that  $e$  *leaves*  $u$  and *enters*  $v$ . Given a vertex  $v$  of  $G$ , we denote by  $d_O^+(v)$  (resp.  $d_O^-(v)$ ) the number of the edges in  $F$  that leave (resp. enter)  $v$  in  $O$ . An *Eulerian graph* is a graph that contains only vertices of even degree, and an *Eulerian orientation* of an Eulerian graph  $G$  is an orientation  $O$  of  $E$  such that  $d_O^+(v) = d_O^-(v)$  for every vertex  $v$  in  $V$ . Note that an Eulerian graph does not need to be connected. Furthermore, we say that a subset  $F \subset E$  is *Eulerian* if  $G[F]$  is Eulerian. We denote by  $G = (A, B; E)$  a bipartite graph  $G$  on vertex classes  $A$  and  $B$ .

We say that a set  $\{H_1, \dots, H_k\}$  of graphs is a *decomposition* of a graph  $G$  if  $\bigcup_{i=1}^k E(H_i) = E$  and  $E(H_i) \cap E(H_j) = \emptyset$  for all  $1 \leq i < j \leq k$ . Let  $\mathcal{H}$  be a family

of graphs. An  $\mathcal{H}$ -decomposition  $\mathcal{D}$  of  $G$  is a decomposition of  $G$  such that each element of  $\mathcal{D}$  is isomorphic to an element of  $\mathcal{H}$ . Furthermore, if  $\mathcal{H} = \{H\}$ , then we say that  $\mathcal{D}$  is an  $H$ -decomposition.

**2.1. Vertex splittings.** Let  $G = (V, E)$  be a graph and  $v$  a vertex of  $G$ . A set  $S_v = \{d_1, \dots, d_{s_v}\}$  of  $s_v$  positive integers is called a *subdegree sequence for  $v$*  if  $d_1 + \dots + d_{s_v} = d_G(v)$ . We say that a graph  $G'$  is obtained by a  $(v, S_v)$ -splitting of  $G$  if  $G'$  is composed of  $G - v$  together with  $s_v$  new vertices  $v_1, \dots, v_{s_v}$  and  $d_G(v)$  new edges such that  $d_{G'}(v_i) = d_i$ , for  $1 \leq i \leq s_v$ , and  $\bigcup_{i=1}^{s_v} N_{G'}(v_i) = N_G(v)$ .

Let  $G$  be a graph and consider a set  $V' = \{v_1, \dots, v_r\}$  of  $r$  vertices of  $G$ . Let  $S_{v_1}, \dots, S_{v_r}$  be subdegree sequences for  $v_1, \dots, v_r$ , respectively. Let  $H_1, \dots, H_r$  be graphs obtained as follows:  $H_1$  is obtained by a  $(v_1, S_{v_1})$ -splitting of  $G$ , the graph  $H_2$  is obtained by a  $(v_2, S_{v_2})$ -splitting of  $H_1$ , and so on, up to  $H_r$ , which is obtained by a  $(v_r, S_{v_r})$ -splitting of  $H_{r-1}$ . We say that each  $H_i$  is an  $\{S_{v_1}, \dots, S_{v_i}\}$ -detachment of  $G$ . Roughly speaking, a detachment of  $G$  is a graph obtained by successive applications of splitting operations on vertices of  $G$ . In Figure 2, the graph  $H$  is an  $\{S_a, S_e\}$ -detachment of  $G$ , where  $S_a = \{2, 3\}$  and  $S_e = \{2, 2, 2\}$ . The next result provides sufficient conditions for the existence of  $2k$ -edge-connected detachments of  $2k$ -edge-connected graphs.

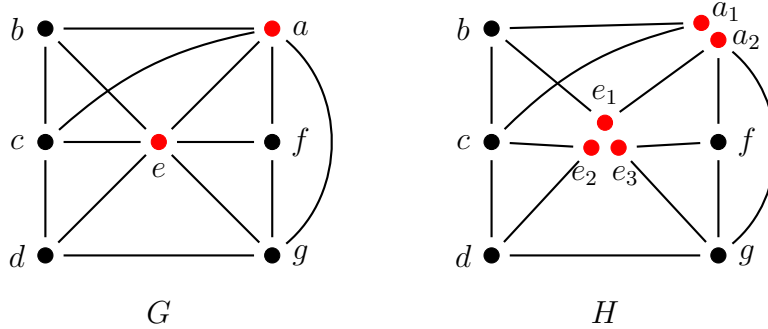


FIGURE 2. A graph  $G$  and a graph  $H$  that is an  $\{S_a, S_e\}$ -detachment of  $G$ .

The next result provides sufficient conditions for the existence of  $2k$ -edge-connected detachments of  $2k$ -edge-connected graphs.

**Lemma 2.1** (Nash–Williams [19]). *Let  $G$  be a  $2k$ -edge-connected graph, where  $k \geq 1$ , and  $V(G) = \{v_1, \dots, v_n\}$ . For every  $v \in V(G)$ , let  $S_v = \{d_1^v, \dots, d_{s_v}^v\}$  be a subdegree sequence for  $v$  such that  $d_i^v \geq 2k$  for  $i = 1, \dots, s_v$ . Then, there exists a  $2k$ -edge-connected  $\{S_{v_1}, \dots, S_{v_n}\}$ -detachment of  $G$ .*

**2.2. Edge liftings.** Let  $G = (V, E)$  be a graph and  $u, v, w$  be distinct vertices of  $G$  such that  $uv, vw \in E$ . The multigraph  $G' = (V, (E \setminus \{uv, vw\}) \cup \{uw\})$  is called a  $uw$ -lifting (or, simply, a *lifting*) at  $v$ . Note that  $G'$  may have parallel edges connecting  $u$  and  $v$ . If for all distinct pairs  $x, y \in V \setminus \{v\}$ , the maximum number of edge-disjoint paths between  $x$  and  $y$  in  $G'$  is the same as in  $G$ , then the lifting at  $v$  is called *admissible*. If  $v$  is a

vertex of degree 2, then the lifting at  $v$  is always admissible. Such a lifting together with the deletion of  $v$  is called a *supression* of  $v$ . The next result is known as Mader's Lifting Theorem.

**Theorem 2.2** (Mader [16]). *Let  $G$  be a multigraph and  $v$  a vertex of  $G$ . If  $v$  is not a cut-vertex,  $d_G(v) \geq 4$ , and  $v$  has at least 2 neighbors, then there exists an admissible lifting at  $v$ .*

The following simple lemma will be useful to apply Mader's Lifting Theorem. In this lemma and thereafter, we denote by  $p_G(x, y)$  the maximum number of edge-disjoint paths between vertices  $x$  and  $y$  in a graph  $G$ .

**Lemma 2.3.** *Let  $G$  be a multigraph and let  $k$  be a positive integer. If  $v$  is a vertex in  $G$  such that  $d(v) < 2k$  and  $p_G(x, y) \geq k$  for any two distinct neighbors  $x$  and  $y$  of  $v$ , then  $v$  is not a cut-vertex.*

**2.3. High edge-connectivity.** If  $G$  is a graph that contains  $2k$  pairwise edge-disjoint spanning trees, then, clearly,  $G$  is  $2k$ -edge-connected. The converse is not true, but as stated in the next theorem, every  $2k$ -edge-connected graph contains  $k$  such trees [18, 26].

**Theorem 2.4** (Nash-Williams [18]; Tutte [26]). *Let  $k$  be a positive integer. If  $G$  is a  $2k$ -edge-connected graph, then  $G$  contains  $k$  pairwise edge-disjoint spanning trees.*

Using Theorem 2.4 and a recent result of Lovász, Thomassen, Wu and Zhang [15], one can prove the following lemma, which enables us to treat highly edge-connected bipartite graphs as regular bipartite graphs. It is a slight generalization of Proposition 2 in [24]. A proof of this lemma is given in [6].

**Lemma 2.5.** *Let  $k \geq 2$  and  $r$  be positive integers. If  $G = (A_1, A_2; E)$  is a  $(6k + 4r - 4)$ -edge-connected bipartite graph and  $|E|$  is divisible by  $k$ , then  $G$  admits a decomposition into two spanning  $r$ -edge-connected graphs  $G_1$  and  $G_2$  such that, the degree in  $G_i$  of each vertex of  $A_i$  is divisible by  $k$ , for  $i = 1, 2$ .*

The following two results on regular multigraphs will be used later (see Figure 1).

**Theorem 2.6** (Von Baebler [27] (see also [1, Theorem 2.37])). *Let  $r \geq 2$  be a positive integer, and  $G$  be an  $(r - 1)$ -edge-connected  $r$ -regular multigraph of even order. Then  $G$  has a 1-factor.*

**Theorem 2.7** (Petersen [20]). *If  $G$  is a  $2k$ -regular multigraph, then  $G$  admits a decomposition into 2-factors.*

The next results are obtained by generalizing a technique used by Bárat and Gerbner [2]. They are useful in the proof of Lemma 2.10, which is used to deal with decompositions into paths of even length.

**Theorem 2.8** (Theorem 20 in [13]). *Let  $m$  be a positive integer. If  $G$  is an  $m$ -edge-connected graph, then  $G$  contains a spanning tree  $T$  such that  $d_T(v) \leq \lceil d_G(v)/m \rceil + 2$  for every vertex  $v$ .*

**Corollary 2.9.** *Let  $m$  be a positive integer. If  $G$  is an  $m$ -edge-connected graph, then  $G$  contains a spanning tree  $T$  such that  $d_T(v) \leq 4 d_G(v)/m$  for every vertex  $v$ .*

*Proof.* From the edge-connectivity of  $G$ , we have  $d_G(v) \geq m$  for every vertex  $v$ . Combining this with Theorem 2.8, we conclude that  $G$  contains a spanning tree  $T$  such that  $d_T(v) \leq \lceil d_G(v)/m \rceil + 2 \leq (d_G(v)/m) + 3 \leq 4 d_G(v)/m$ .  $\square$

**Lemma 2.10.** *Let  $k, m$  and  $r$  be positive integers, and let  $G = (A, B; E)$  be a bipartite graph. If  $G$  is  $8m \lceil (k+r)/k \rceil$ -edge-connected and, for every  $v \in A$ ,  $d_G(v)$  is divisible by  $k+r$ , then  $G$  admits a decomposition into spanning graphs  $G_k$  and  $G_r$  such that  $G_k$  is  $m$ -edge-connected and, for every vertex  $v \in A$ , we have  $d_{G_k}(v) = \frac{k}{k+r} d_G(v)$  and  $d_{G_r}(v) = \frac{r}{k+r} d_G(v)$ .*

*Proof.* Let  $k, m, r$  and  $G = (A, B; E)$  be as in the hypothesis of the lemma. Since  $G$  is  $8m \lceil (k+r)/k \rceil$ -edge-connected, by Theorem 2.4, we conclude that  $G$  contains at least  $4m \lceil (k+r)/k \rceil$  pairwise edge-disjoint spanning trees. Now, partition the set of these  $4m \lceil (k+r)/k \rceil$  spanning trees into  $m$  sets, say  $\mathcal{T}_1, \dots, \mathcal{T}_m$ , of  $4 \lceil (k+r)/k \rceil$  spanning trees each, and let  $G_i = \bigcup_{T \in \mathcal{T}_i} T$ , for  $i = 1, \dots, m$ .

Clearly,  $G_i$  is  $4 \lceil (k+r)/k \rceil$ -edge-connected. By Corollary 2.9,  $G_i$  contains a spanning tree  $T_i$  such that, for every  $v \in V(G_i)$ ,

$$d_{T_i}(v) \leq \frac{1}{\lceil (k+r)/k \rceil} d_{G_i}(v) \leq \left( \frac{k}{k+r} \right) d_{G_i}(v).$$

Let  $G' = \bigcup_{i=1}^m T_i$ . Clearly,  $G'$  is  $m$ -edge-connected. Note that, for every  $v \in V(G)$ ,

$$d_{G'}(v) = \sum_{i=1}^m d_{T_i}(v) \leq \left( \frac{k}{k+r} \right) \sum_{i=1}^m d_{G_i}(v) \leq \left( \frac{k}{k+r} \right) d_G(v).$$

Let  $G_k$  be the bipartite graph obtained from  $G'$  by adding, for every vertex  $v$  in  $A$ , exactly  $((k/(k+r))d_G(v) - d_{G'}(v))$  edges of  $G - E(G')$  that are incident to  $v$  (note that  $((k/(k+r))d_G(v))$  is an integer). Therefore, every vertex  $v \in A$  has degree exactly  $((k/(k+r))d_G(v))$  in  $G_k$ . To conclude the proof, take  $G_r = G - E(G_k)$ .  $\square$

### 3. THE DISENTANGLING LEMMA

Our aim in this section is to prove a result, Lemma 3.11, which guarantees that, given a special trail decomposition of a graph  $G$ , it is possible to switch edges of the elements of this decomposition and construct a path decomposition of  $G$ . For that, we introduce the concept of *trackings* of a trail: they are important to specify the order in which the vertices of a trail are visited.

We came to know recently that the technique introduced in this section generalizes the one presented by Kouider and Lonc [14] for decompositions of girth-restricted even regular graphs into paths. Here, we manage to overcome this girth condition, by requiring a sufficiently high minimum degree.

**3.1. Trails, trackings and augmenting sequences.** A *trail* is a graph  $T$  for which there is a sequence  $B = x_0 \cdots x_\ell$  of its vertices (possibly with repetitions) such that  $E(T) = \{x_i x_{i+1} : 0 \leq i \leq \ell - 1\}$ ; such a sequence is called a *tracking* of  $T$ , and we say that  $T$  is the trail *induced* by the tracking  $B$ . Note that a path admits only two possible trackings, while a cycle of length  $\ell$  admits  $2\ell$  trackings. The vertices  $x_0$  and  $x_\ell$  are called *end-vertices* of  $B$ .

Given a tracking  $B = x_0 \cdots x_\ell$ , we denote by  $B^-$  the tracking  $x_\ell \cdots x_0$ , and, to ease notation, we denote by  $V(B)$  and  $E(B)$  the sets  $\{x_0, \dots, x_\ell\}$  of vertices and  $\{x_i x_{i+1} : 0 \leq i \leq \ell - 1\}$  of edges of  $B$ , respectively. Moreover, we denote by  $\bar{B}$  the trail  $(V(B), E(B^-))$ .

It will be convenient to say that a tracking  $B = x_0 \cdots x_\ell$  *traverses* the vertices  $x_0, \dots, x_\ell$  and the edges  $x_0 x_1, \dots, x_{\ell-1} x_\ell$  (in this order), and that  $x_0 x_1$  is the *starting edge* of  $B$  and  $x_{\ell-1} x_\ell$  is the *ending edge* of  $B$ , or that  $B$  *starts* with  $x_0 x_1$  and *ends* with  $x_{\ell-1} x_\ell$ .

We say that a trail  $T$  is a *vanilla trail* if there is a tracking  $x_0 x_1 \cdots x_\ell$  of  $T$  such that  $x_1 \cdots x_{\ell-1}$  induces a path in  $G$ . A tracking that induces a vanilla trail is also called a *vanilla tracking*. (See Figure 3.)

If a vanilla trail contains  $\ell$  edges, then we say that it is a *vanilla  $\ell$ -trail*. A set  $\mathcal{B}$  of pairwise edge-disjoint trackings of vanilla  $\ell$ -trails of a graph  $G$  is an  *$\ell$ -tracking decomposition* of  $G$  if  $\bigcup_{B \in \mathcal{B}} E(B) = E(G)$ , i.e.,  $\{\bar{B} : B \in \mathcal{B}\}$  is a decomposition of  $G$  into vanilla  $\ell$ -trails. If every element of  $\mathcal{B}$  induces an  $\ell$ -path, then we say that  $\mathcal{B}$  is an  *$\ell$ -path tracking decomposition*. We may omit the length  $\ell$ , when it is clear from the context. We note that if  $B_i$  and  $B_j$  are trackings of a tracking decomposition  $\mathcal{B}$  such that  $E(B_i) \cap E(B_j) \neq \emptyset$ , then  $\bar{B}_i = \bar{B}_j$  (that is,  $B_i$  and  $B_j$  induce the same vanilla trail).

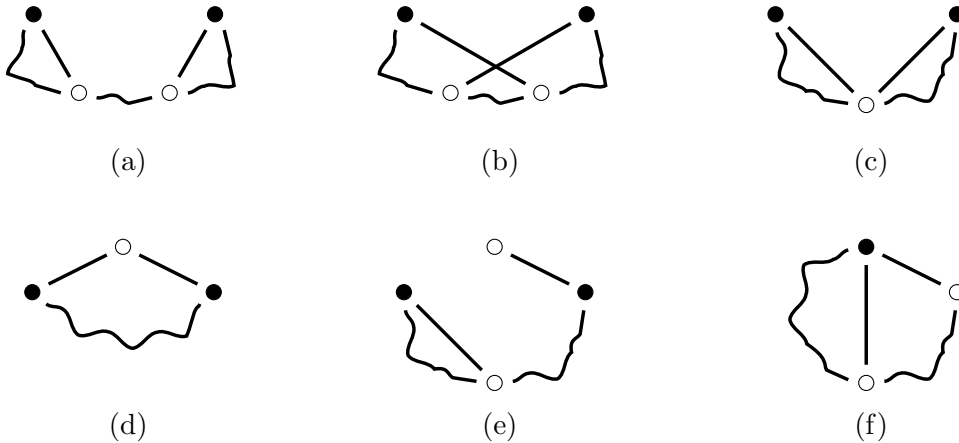


FIGURE 3. Examples of vanilla trails.

The concept of augmenting sequence (Definition 3.1) is central in this section. Before presenting it, we give a motivation for it.

For every vanilla trail  $T$  of  $G$ , let  $\tau(T)$  be the number of end-vertices of  $T$  with degree greater than 1. Let  $\mathcal{D}$  be a decomposition of  $G$  into vanilla  $\ell$ -trails that minimizes  $\tau(\mathcal{D}) = \sum_{T \in \mathcal{D}} \tau(T)$ . If  $\tau(\mathcal{D}) = 0$ , then  $\mathcal{D}$  is an  $\ell$ -path decomposition. So, let us assume that  $\tau(\mathcal{D}) > 0$ . Moreover, suppose that  $\mathcal{D}$  has the following property: for every  $T$  in  $\mathcal{D}$  and every vertex  $v$  of  $T$ , there is a trail  $T'$  containing an edge  $vu$ , such that  $u \notin V(T)$  and  $u$  is an end-vertex of  $T'$ .

Since  $\tau(\mathcal{D}) > 0$ , there is a vanilla trail  $T_0$  in  $\mathcal{D}$  that is not a path. Let  $x$  be an end-vertex of  $T_0$  of degree greater than 1, and let  $C$  be a cycle in  $T_0$  that contains  $x$ . Consider a neighbour  $v$  of  $x$  in  $C$ , and let  $T_1$  be an element of  $\mathcal{D}$  that contains an edge  $vu$ , such that  $u \notin V(T_0)$  and  $u$  is an end-vertex of  $T_1$ , as supposed above. Now, let  $T'_0 = T_0 - vx + vu$ ,  $T'_1 = T_1 - vu + vx$ , and put  $\mathcal{D}' = \mathcal{D} - T_0 - T_1 + T'_0 + T'_1$ . We have  $\tau(T'_0) = \tau(T_0) - 1$ . If  $\tau(T'_1) \leq \tau(T_1)$ , then  $\mathcal{D}'$  is a decomposition of  $G$  into vanilla  $\ell$ -trails such that  $\sum_{T \in \mathcal{D}'} \tau(T) < \sum_{T \in \mathcal{D}} \tau(T)$ , a contradiction to the minimality of  $\tau(\mathcal{D})$ . Therefore,  $\tau(T'_1) = \tau(T_1) + 1$  and  $T'_1$  contains a cycle  $C'$  that contains  $xv$ . Now, we consider a neighbour  $v'$  of  $x$  in  $C'$  such that  $v' \neq v$ , and we repeat this operation as long as necessary considering  $T'_1$  and  $v'$  instead of  $T_0$  and  $v$ .

We show that, under some assumptions, after repeating this operation a finite number of times, we obtain a better decomposition (an  $\ell$ -tracking decomposition in which there are more trackings inducing paths than the previous one). The next definition formalizes which properties the sequence of trails must satisfy to guarantee this improvement.

To formalize the ideas mentioned before, let us introduce some concepts. Let  $\mathcal{B}$  be an  $\ell$ -tracking decomposition of a graph  $G$ , and let  $S = B_1 B_2 \cdots B_r$  be a sequence of (not necessarily distinct) trackings of trails of  $G$ , where  $B_i = b_0^i b_1^i \cdots b_\ell^i$ , for  $i = 1, \dots, r$ . We say that  $S$  is a  $\mathcal{B}$ -sequence if  $B_i \in \mathcal{B}$  or  $B_i^- \in \mathcal{B}$ , for  $i = 1, \dots, r$ .

In what follows, we shall be interested in such  $\mathcal{B}$ -sequences  $S = B_1 B_2 \cdots B_r$ , in which the vertex  $b_0^1$ , the first vertex of  $B_1$ , plays an important role. We require that each element  $B_i$  of  $S$ , except the last one (the tracking  $B_r$ ), contains the vertex  $b_0^1$ . We denote by  $s(i)$  the smallest positive index such that  $b_{s(i)}^i = b_0^1$ . As we need to refer frequently to the vertex  $b_{s(i)-1}^i$  (the vertex of  $B_i$  that is traversed before  $b_0^1$ ) and the vertex  $b_{s(i)+1}^i$  (the vertex of  $B_i$  that is traversed after  $b_0^1$ ), for ease of notation, we also denote them by  $b_*^i$  and  $b_+^i$  (that is,  $b_*^i := b_{s(i)-1}^i$  and  $b_+^i := b_{s(i)+1}^i$ ), respectively. In this context, we also give names to two special edges of each  $B_i$ ; these are  $e_i := b_*^i b_0^1$  (the edge traversed by  $B_i$  “to enter”  $b_0^1$ ), and  $f_i := b_0^i b_1^i$  (the starting edge of  $B_i$ ). See Figure 4.



**Definition 3.1.** Let  $\ell$  and  $r \geq 2$  be positive integers and let  $\mathcal{B}$  be an  $\ell$ -tracking decomposition of a graph  $G$ . Let  $S = B_1 B_2 \cdots B_r$  be a  $\mathcal{B}$ -sequence, where  $B_i = b_0^i b_1^i \cdots b_\ell^i$  for  $i = 1 \dots, r$ . We say that  $S$  is an augmenting sequence of  $\mathcal{B}$  if

- (i1)  $\bar{B}_1$  is not a path, and  $d_{\bar{B}_1}(b_0^1) > 1$ ,  $b_0^2 \notin V(B_1)$ ;
- (i2)  $b_1^r = b_*^{r-1}$ ;

and for  $i = 2, \dots, r-1$ , the following holds:

- (ii)  $B_i$  contains  $b_0^1$  and  $b_1^i = b_*^{i-1}$ ;
- (iii) if  $\bar{B}_i \neq \bar{B}_h$  for every  $h < i$ , then  $b_0^{i+1} \notin V(B_i)$ ;
- (iv) if  $\bar{B}_i = \bar{B}_1$ , then  $b_0^{i+1} \notin V(\bar{B}_1 - e_1 + f_2) = V(\bar{B}_i) \cup \{b_0^2\}$ .
- (v) if  $\bar{B}_i = \bar{B}_h$  for some  $1 < h < i$ , then  $b_0^{i+1} \notin V(\bar{B}_h - f_h + e_{h-1} - e_h + f_{h+1})$ .

If, in addition,  $b_0^1 \notin V(B_r)$ , then we say that  $S$  is a full-augmenting sequence.

We note that whenever we delete edges of a trail (as in (iv) and (v)), we also remove the isolated vertices that may result after the edge deletions. We also observe that (iv) implies that if  $\bar{B}_i = \bar{B}_1$  then  $B_{i+1} \neq B_i$  (this will be used later).

The main idea behind our central result is that, given a certain tracking decomposition, if we can find a full-augmenting sequence, then we can find a better decomposition. Thus, the conditions stated in Definition 3.1 have the purpose of allowing interchanging of edges of the elements of a full-augmenting sequence. Such an interchange will be performed starting from the first element  $B_1$  and then going from  $B_i$  to  $B_{i+1}$ . If the elements are

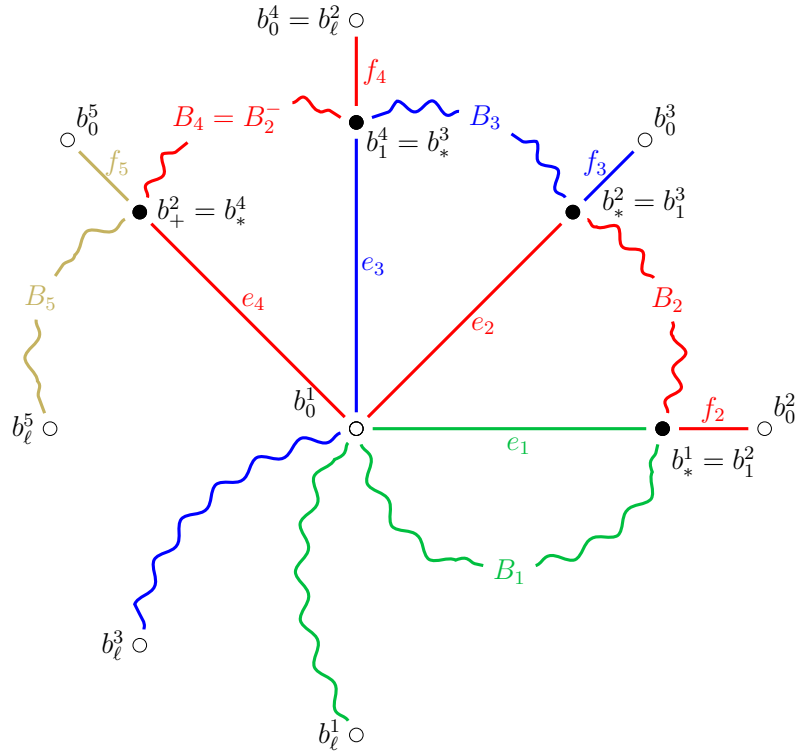


FIGURE 4. An augmenting sequence  $S = B_1 B_2 B_3 B_4 B_5$ , where  $B_4 = B_2^-$ .

all distinct, then the simple interchange we have mentioned in the motivation suffices, as long as the items (i)–(iii) are satisfied. But, as the trail corresponding to some trackings may repeat, we need the conditions stated in (iv) and (v). Note that item (iv) requires that if  $\bar{B}_i = \bar{B}_1$ , then the initial vertex of  $B_{i+1}$  does not belong to the trail corresponding to the tracking to which  $B_1$  was transformed (that is, the trail  $\bar{B}_1 - e_1 + f_2$ ). Item (v) requires that if  $\bar{B}_i = \bar{B}_h$  for some  $1 < h < i$ , then the initial vertex of  $B_{i+1}$  does not belong to the trail corresponding to the tracking to which  $B_h$  was transformed. In this case, since  $1 < h < i$ , the original tracking  $B_h$  has suffered two transformations. (Suppose  $r > 2$ .)  $B_2$  suffers a first transformation (because of  $B_1$ ), but then, the transformed  $B_2$  plays the role of the original  $B_1$ , and so it is again transformed because of  $B_3$ . Thus, the condition stated in item (v) reflects this double transformation suffered by  $B_h$ . To understand this idea, consider the augmenting sequence shown in Figure 4, where  $B_4 = B_2^-$  (that is,  $\bar{B}_4 = \bar{B}_2$ ), and see the step-by-step transformations shown in Figure 5.

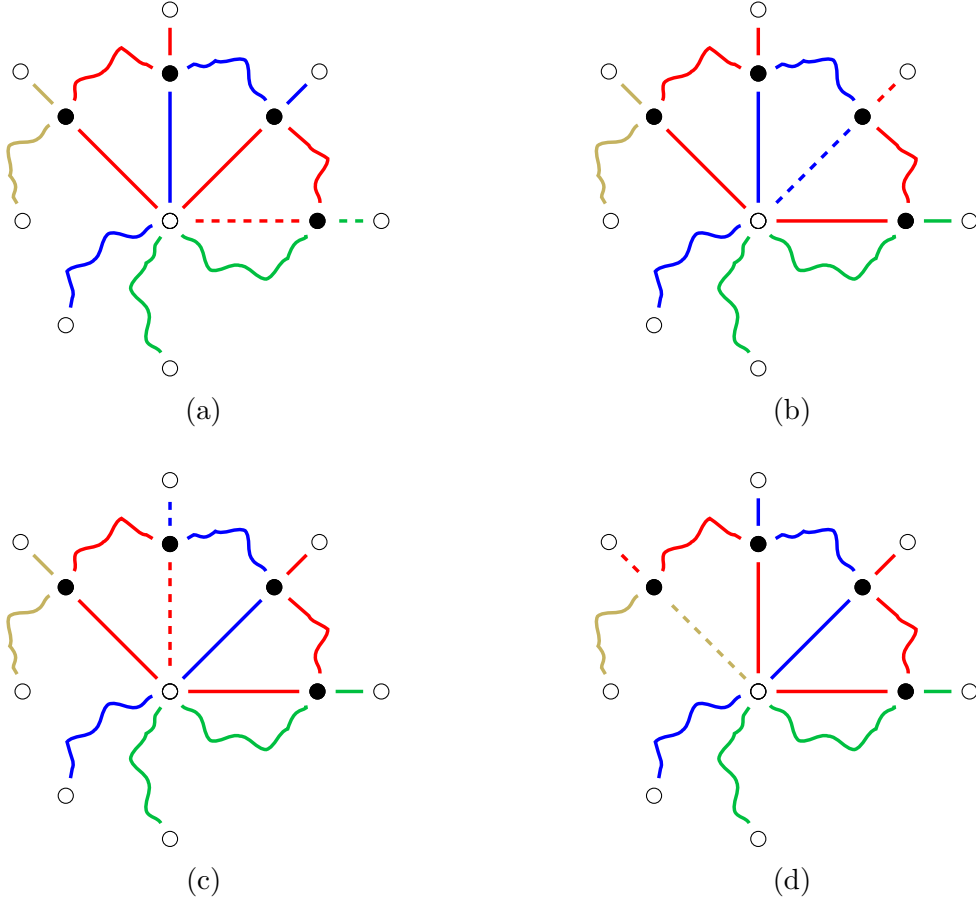


FIGURE 5. Illustration of how to deal with the full-augmenting sequence in Figure 4. In each step, the dashed edges are those that are switched.

As we will see, full-augmenting sequences of a tracking decomposition  $\mathcal{B}$  have a finite number of elements. To prove this (Corollary 3.3), we show first the following result.

**Lemma 3.2.** *Let  $\ell$  and  $r \geq 2$  be positive integers and let  $\mathcal{B}$  be an  $\ell$ -tracking decomposition of a graph  $G$ . If  $S = B_1 B_2 \cdots B_r$  is an augmenting sequence of  $\mathcal{B}$ , where  $B_i = b_0^i b_1^i \cdots b_\ell^i$  for  $i = 1, \dots, r$ , then  $b_*^i \neq b_*^j$  for every  $i, j$  with  $1 \leq i < j \leq r - 1$ .*

*Proof.* Let  $\ell, r, \mathcal{B}$  and  $S = B_1 B_2 \cdots B_r$  be as in the hypothesis of the lemma. We want to prove that  $\{b_*^1, b_*^2, \dots, b_*^{r-1}\}$  is a set of distinct elements.

*Claim A:*  $b_*^j \neq b_*^1$ , for  $j = 2, \dots, r - 1$ .

For  $j = 2$  the result is immediate. Indeed, recall that  $e_i = b_*^i b_0^1 \in E(B_i)$ . If  $b_*^2 = b_*^1$  then  $e_1 = e_2$ , that is,  $\bar{B}_1$  and  $\bar{B}_2$  have a common edge. But then,  $\bar{B}_1 = \bar{B}_2$ , a contradiction (to Definition 3.1 (i1)). Now suppose  $j \geq 3$ . Take such a smallest index  $j$  for which  $b_*^j = b_*^1$ . As in the previous case, we conclude that  $\bar{B}_j = \bar{B}_1$ . Since  $S$  is a  $\mathcal{B}$ -sequence, either  $B_j = B_1^-$  or  $B_j = B_1$ . If  $B_j = B_1^-$ , then  $b_*^j = b_+^1$ . But  $b_+^1 \neq b_*^1$ . Thus,  $b_*^j \neq b_*^1$ , a contradiction. If  $B_j = B_1$ , then  $b_0^j = b_0^1$  and  $f_j = f_1$ . But  $b_1^j = b_*^{j-1}$  (by Definition 3.1 (ii)). Hence,  $f_j = b_0^j b_1^j = b_0^1 b_*^{j-1} = e_{j-1}$ , that is,  $f_j \in B_{j-1}$ . Since  $f_j = f_1 \in B_1$ , we conclude that  $\bar{B}_{j-1} = \bar{B}_1$ . Thus,  $\bar{B}_{j-1} = \bar{B}_1 = \bar{B}_j$ , that is,  $\bar{B}_{j-1} = \bar{B}_j$ , a contradiction (see the observation after Definition 3.1).

*Claim B:*  $b_*^i \neq b_*^j$  for every  $i, j$  with  $2 \leq i < j \leq r - 1$ .

Suppose that this does not hold. Let  $i$  be the smallest integer such that there exists  $j > i$  such that  $b_*^i = b_*^j$ . In this case,  $e_i = b_*^i b_0^1 = b_*^j b_0^1 = e_j$ , and thus  $\bar{B}_i = \bar{B}_j$ . Hence, either  $B_j = B_i$  or  $B_j = B_i^-$ . If  $B_j = B_i^-$ , then  $b_*^j = b_+^i \neq b_*^i$ , a contradiction. If  $B_j = B_i$ , then  $b_0^j = b_0^i$ . Hence,  $f_j = b_0^j b_1^j = b_0^i b_*^{j-1}$  and  $f_i = b_0^i b_1^i = b_0^i b_*^{i-1}$ . Since  $f_j = f_i$ , we conclude that  $b_*^{j-1} = b_*^{i-1}$ , a contradiction to the choice of  $i$ .  $\square$

**Corollary 3.3.** *Let  $\ell$  and  $r \geq 2$  be positive integers and let  $\mathcal{B}$  be an  $\ell$ -tracking decomposition of a graph  $G$ . If  $S = B_1 B_2 \cdots B_r$  is an augmenting sequence of  $\mathcal{B}$ , then each  $B_i$  occurs at most once in  $S$ . Furthermore, if  $\bar{B}_i = \bar{B}_j$  for some pair  $i, j$  with  $1 \leq i < j \leq r - 1$ , then  $B_j = B_i^-$ .*

*Proof.* Let  $\ell, r, \mathcal{B}$  and  $S = B_1 B_2 \cdots B_r$  be as in the hypothesis of the corollary. Let  $B_i = b_0^i b_1^i \cdots b_\ell^i$ , for  $i = 1, \dots, r$ . Suppose, for a contradiction, that  $B_i = B_j$  for some pair  $i, j$  with  $1 \leq i < j \leq r - 1$ . In this case,  $e_i = e_j$ , and therefore,  $b_*^i = b_*^j$ , a contradiction to Lemma 3.2. Now, since  $S$  is a  $\mathcal{B}$ -sequence and  $B_i \neq B_j$ , if  $\bar{B}_i = \bar{B}_j$ , then  $B_j = B_i^-$ .  $\square$

Corollary 3.3 implies that any augmenting sequence of an  $\ell$ -tracking decomposition is finite.

**3.2. Hanging edges and complete tracking decomposition.** All concepts defined in this subsection refers to a tracking decomposition  $\mathcal{B}$  of a graph  $G$ . We recall that any tracking in  $\mathcal{B}$  has exactly two end-vertices, even if they coincide. For  $B$  in  $\mathcal{B}$ , we denote by  $\tau(B)$  the number of end-vertices of  $B$  that have degree greater than 1 in  $\bar{B}$ . Thus,  $\tau(B) = 0$  if and only if  $\bar{B}$  is a path. We observe that the same notation is used for trails (as the meaning for both coincides). Let  $\tau(\mathcal{B}) = \sum_{B \in \mathcal{B}} \tau(B)$ .

Let  $uv$  be an edge of  $G$ , and let  $B$  be the element of  $\mathcal{B}$  that contains  $uv$ . If  $B = x_0x_1 \cdots x_\ell$  with either  $x_0 = u$  and  $x_1 = v$ , or  $x_\ell = u$  and  $x_{\ell-1} = v$ , then we say that  $uv$  is a *pre-hanging* edge at  $v$  in  $\mathcal{B}$ . If, additionally,  $d_{\bar{B}}(u) = 1$ , then we say that  $uv$  is a *hanging* edge at  $v$  in  $\mathcal{B}$ . We denote by  $\text{preHang}(v, \mathcal{B})$  (resp.  $\text{hang}(v, \mathcal{B})$ ) the number of pre-hanging (resp. hanging) edges at  $v$  in  $\mathcal{B}$ . Let  $k$  be a positive integer. We say that  $\mathcal{B}$  is *k-pre-complete* if  $\text{preHang}(v, \mathcal{B}) > k$  for every  $v$  in  $V(G)$ . If  $\text{hang}(v, \mathcal{B}) > k$  for every  $v$  in  $V(G)$ , then we say that  $\mathcal{B}$  is *k-complete*.

For  $v$  in  $V(G)$ , let  $\mathcal{B}_{\text{odd}}(v)$  be the number of elements  $B$  of  $\mathcal{B}$  such that  $d_{\bar{B}}(v)$  is odd, and let  $\mathcal{B}_{\text{even}}(v)$  be the number of elements  $B = x_0 \cdots x_\ell$  in  $\mathcal{B}$  such that  $x_0 = x_\ell = v$ . Furthermore, define  $\mathcal{B}(v) = \mathcal{B}_{\text{odd}}(v) + 2\mathcal{B}_{\text{even}}(v)$ . One can see  $\mathcal{B}(v)$  as the number of edges of  $G$  incident to  $v$  that are starting edges of trackings in  $\mathcal{B}$  that start at  $v$ , or ending edges of trackings in  $\mathcal{B}$  that end at  $v$ . We note that if  $\mathcal{B}$  is an  $\ell$ -tracking decomposition of  $G$ , then  $\sum_{v \in V(G)} \mathcal{B}(v) = 2|\mathcal{B}| = 2|E(G)|/\ell$ , because each element of  $\mathcal{B}$  has exactly two end-vertices (counted with their multiplicities). The next lemma is the main tool in the proof of the Disentangling Lemma (Lemma 3.11).

**Lemma 3.4.** *Let  $k$  and  $\ell$  be positive integers and let  $\mathcal{B}$  be a  $k$ -complete  $\ell$ -tracking decomposition of a graph  $G$ . If  $\mathcal{B}$  contains a full-augmenting sequence, then there is an  $\ell$ -tracking decomposition  $\mathcal{B}'$  of  $G$  such that the following holds.*

- $\tau(\mathcal{B}') < \tau(\mathcal{B})$ ;
- $\mathcal{B}'(v) = \mathcal{B}(v)$  for every  $v \in V(G)$ ;
- $\mathcal{B}'$  is  $k$ -complete.

*Proof.* Let  $k$ ,  $\ell$  and  $\mathcal{B}$  be as in the hypothesis of the lemma. Suppose that  $S = B_1 \cdots B_r$  is a full-augmenting sequence of  $\mathcal{B}$ , where  $S = B_1 \cdots B_r$ , and  $B_i = b_0^i b_1^i \cdots b_\ell^i$  for  $i = 1, \dots, r$ .

The proof is by induction on the number of elements of  $S$ , denoted by  $|S|$ . Note that by the definition of full-augmenting sequence, we have  $b_0^1 \notin V(B_r)$ . Therefore,  $|S| = r > 1$ .

Suppose  $|S| = 2$ . Since  $S$  is a full-augmenting sequence,  $b_0^1 \notin V(B_2)$  and, by item (ii) of Definition 3.1,  $b_1^2 = b_*^1$ . Let  $\bar{B}'_1 = \bar{B}_1 - e_1 + f_2$  and  $\bar{B}'_2 = \bar{B}_2 - f_2 + e_1$ . That is,  $\bar{B}'_1$  and  $\bar{B}'_2$  are obtained from  $\bar{B}_1$  and  $\bar{B}_2$  by interchanging the edges  $e_1$  and  $f_2$ . Then we consider the following trackings corresponding to these trails:  $B'_1 = b_0^2 X^{-1} b_+^1 \cdots b_\ell^1$ , where  $X = b_0^1 b_1^1 \cdots b_*^1$ ; and  $B'_2 = b_0^1 b_1^2 b_2^2 \cdots b_\ell^2$ . It is easy to see that  $B'_1$  and  $B'_2$  are  $\ell$ -trackings of  $G$ , and furthermore,  $\bar{B}_1 \cup \bar{B}_2 = \bar{B}'_1 \cup \bar{B}'_2$ .

Let  $\mathcal{B}' = \mathcal{B} - B_1 - B_2 + B'_1 + B'_2$ . Clearly,  $\mathcal{B}'$  is an  $\ell$ -tracking decomposition of  $G$ . By items (i1) and (iii) of Definition 3.1,  $d_{\bar{B}_1}(b_0^1) > 1$  and  $b_0^2 \notin V(B_1)$ , from where we conclude that  $d_{\bar{B}'_1}(b_0^2) = 1$  and  $\tau(B'_1) \leq \tau(B_1) - 1$ . Since  $b_0^1 \notin V(B_2)$ , we have  $d_{\bar{B}'_2}(b_0^1) = 1$ . Thus,  $\tau(B'_2) \leq \tau(B_2)$ , and therefore the following inequality holds.

$$\tau(\mathcal{B}') = \tau(\mathcal{B}) - \tau(B_1) - \tau(B_2) + \tau(B'_1) + \tau(B'_2) < \tau(\mathcal{B}).$$

It remains to prove (for  $|S| = 2$ ) that  $\mathcal{B}'(v) = \mathcal{B}(v)$  for every  $v \in V(G)$ , and that  $\mathcal{B}'$  is  $k$ -complete.

**Claim 3.5.**  $\mathcal{B}'(v) = \mathcal{B}(v)$  for every  $v \in V(G)$ .

*Proof.* Given  $v \in V(G)$  and a set  $\mathcal{T} \subset \mathcal{B}$ , define  $\mathcal{B}_{\text{odd}}|_{\mathcal{T}}(v)$  as the number of elements  $B \in \mathcal{T}$  such that  $d_{\bar{B}}(v)$  is odd, and define  $\mathcal{B}_{\text{even}}|_{\mathcal{T}}$  as the number of elements  $B = x_0 \cdots x_\ell$  of  $\mathcal{T}$  such that  $x_0 = x_\ell = v$ . Furthermore, let  $\mathcal{B}|_{\mathcal{T}}(v) = \mathcal{B}_{\text{odd}}|_{\mathcal{T}}(v) + 2\mathcal{B}_{\text{even}}|_{\mathcal{T}}(v)$ .

Let  $B_{\text{vert}} = \{b_0^1, b_0^2, b_1^2\}$ . Clearly,  $\mathcal{B}(v) = \mathcal{B}'(v)$  for every  $v \notin B_{\text{vert}}$ . Let  $\mathcal{T} = \{B_1, B_2\}$  and  $\mathcal{T}' = \{B'_1, B'_2\}$ . To prove that  $\mathcal{B}(v) = \mathcal{B}'(v)$  holds also for vertices  $v \in B_{\text{vert}}$ , it is enough to show that  $\mathcal{B}|_{\mathcal{T}}(v) = \mathcal{B}'|_{\mathcal{T}'}(v)$ , because we already know that  $\mathcal{B}|_{\mathcal{B} \setminus \mathcal{T}}(v) = \mathcal{B}'|_{\mathcal{B}' \setminus \mathcal{T}'}(v)$ . Recall that  $\mathcal{B}|_{\mathcal{T}}(v)$  (resp.  $\mathcal{B}'|_{\mathcal{T}'}(v)$ ) is the number edges of  $G$  that are starting edges of elements in  $\mathcal{T}$  (resp.  $\mathcal{T}'$ ) that start at  $v$ , or ending edges of elements in  $\mathcal{T}$  (resp.  $\mathcal{T}'$ ) that end at  $v$ . First, note that  $\mathcal{B}|_{\mathcal{T}}(b_0^1) = \mathcal{B}'|_{\mathcal{T}'}(b_0^1)$ . Indeed, the edge  $f_1 = b_0^1 b_1^1$  is the starting edge of  $B_1$ , but it is not a starting edge of neither  $B'_1$  or  $B'_2$ ; but, on the other hand, the starting edge of  $B'_2$  starts at  $b_0^1$ . Thus, the number of starting edges that starts at  $b_0^1$  is the same in  $\mathcal{B}$  and in  $\mathcal{B}'$ . In terms of ending edges that end at  $b_0^1$ , the same happens: if  $e_1 = b_1^1 b_0^1$  is an ending edge of  $B_1$ , then the ending edge of  $B'_1$  (which is the reverse of  $f_1$ ) also ends at  $b_0^1$ , and if  $e_1$  is not an ending edge of  $B_1$ , then neither  $B'_1$  or  $B'_2$  has an ending edge incident to  $b_0^1$ . It is easy to see that  $\mathcal{B}|_{\mathcal{T}}(b_1^2) = \mathcal{B}'|_{\mathcal{T}'}(b_1^2)$ , as  $b_1^2$  is an internal vertex of all trackings under analysis. Also,  $\mathcal{B}|_{\mathcal{T}}(b_0^2) = \mathcal{B}'|_{\mathcal{T}'}(b_0^2)$ , as the starting edge  $f_2 = b_0^2 b_1^2$  of  $B_2$  becomes the starting edge of  $B'_1$ , and no other change occurs in terms of ending edges at  $b_0^2$ .  $\square$

**Claim 3.6.**  $\mathcal{B}'$  is  $k$ -complete.

*Proof.* Let us prove that  $\text{hang}(v, \mathcal{B}') > k$  for every  $v \in V(G)$ . Note that if  $v \neq b_1^2$ , then the hanging edges at  $v$  in  $\mathcal{B}'$  are the same hanging edges at  $v$  in  $\mathcal{B}$ . Let  $E_{\text{hang}}$  be the set of hanging edges at  $b_1^2$  in  $\mathcal{B}$ . Since  $d_{\bar{B}_1}(b_1^2) > 1$  (by Definition 3.1 (i1)), we know that  $e_1 = b_0^1 b_1^2 \notin E_{\text{hang}}$ . The set of hanging edges at  $b_1^2$  in  $\mathcal{B}'$  is  $E_{\text{hang}} \cup \{b_0^1 b_1^2, b_0^2 b_1^2\}$  because  $d_{\bar{B}'_1}(b_1^2) = 1$  and  $d_{\bar{B}'_2}(b_1^2) = 1$ . Then,  $\text{hang}(b_1^2, \mathcal{B}') > \text{hang}(b_1^2, \mathcal{B}) > k$ . Therefore,  $\mathcal{B}'$  is  $k$ -complete.  $\square$

In the rest of the proof we assume that  $|S| = r > 2$ . Suppose that the lemma holds when  $\mathcal{B}$  contains a full-augmenting sequence  $S'$  with length  $r - 1$ .

Since  $|S| > 2$ , by item (ii) of Definition 3.1 we have  $b_1^2 = b_*^1$  and  $b_{s(2)}^2 = b_0^1$  where  $s(2) \geq 3$ . Now, consider the trackings  $B'_1$  and  $B'_2$  that we have defined in the proof for the case  $|S| = 2$ . Let  $\mathcal{B}''$  be the  $\ell$ -tracking decomposition as we have defined in that case (which we called  $\mathcal{B}'$ ), that is,  $\mathcal{B}'' = \mathcal{B} - B_1 - B_2 + B'_1 + B'_2$ . In the case  $|S| = 2$ , we had  $b_0^1 \notin V(B_2)$ , but now we have that  $b_0^1 \in V(B_2)$ , thus, in this case we can only conclude that  $\tau(\mathcal{B}'') \leq \tau(\mathcal{B})$ . See Figure 5a.

The next step is to prove that  $\mathcal{B}''(v) = \mathcal{B}(v)$  for all  $v \in V(G)$ . The proof follows analogously to the proof we have presented for the case  $|S| = 2$ .

Now we will prove that  $\mathcal{B}''$  is  $k$ -complete. Note that if  $v \neq b_1^2$ , then the hanging edges at  $v$  in  $\mathcal{B}$  are the same hanging edges at  $v$  in  $\mathcal{B}''$ . Now, let  $E_{\text{hang}}(b_1^2)$  be the set of hanging

edges at  $b_1^2$  in  $\mathcal{B}$ . Then the set of hanging edges at  $b_1^2$  in  $\mathcal{B}''$  is  $E_{\text{hang}}(b_1^2) \cup \{b_1^2 b_0^2\}$  because  $d_{\bar{B}_1'}(b_0^2) = 1$ . Then,  $\text{hang}(b_1^2, \mathcal{B}'') \geq \text{hang}(b_1^2, \mathcal{B}) > k$ . Therefore,  $\mathcal{B}''$  is  $k$ -complete.

Since  $S$  is an augmenting sequence of  $\mathcal{B}$ , by Corollary 3.3, every  $\bar{B}_i$  appears at most twice in  $S$  and if  $\bar{B}_i = \bar{B}_j$  for  $1 \leq i < j \leq r$ , then,  $B_j = B_i^-$ . Let  $S' = C_2 C_3 \cdots C_r$ , where,  $C_2 = B_2'$  and for  $3 \leq i \leq r$ , we have

$$C_i = \begin{cases} B_1'^- & \text{if } B_i = B_1^-; \\ B_2'^- & \text{if } B_i = B_2^-; \\ B_i & \text{otherwise.} \end{cases}$$

We shall prove that  $S'$  is a full-augmenting sequence. For that, we shall check each of the items of Definition 3.1. Before, we make some observations: we also denote by  $s(i)$  the smallest index such that  $c_0^2 = c_{s(i)}^i = b_0^1$ , for  $2 \leq i \leq r$ . The vertex  $c_j^i$  is the same as  $b_j^i$  for  $i = 3, \dots, r$  and  $j = 0, \dots, s(i)$ . We denote by  $e_i^*$  and  $f_i^*$  the edges of  $C_i$  that correspond to  $e_i$  and  $f_i$  defined for  $B_i$ , that is,  $e_i^* = c_i^i b_0^1$  and  $f_i^* = c_0^i c_1^i$ .

**Item (i):**  $\bar{C}_2$  is not a path,  $d_{\bar{C}_2}(c_0^2) > 1$  and  $c_0^3 \notin V(C_2)$ .

Since  $C_2 = B_2'$ , we have  $c_0^2 = b_0^1$ . Moreover, since  $\bar{B}_2' = \bar{B}_2 - f_2 + e_1$ , the edges  $e_1$  and  $e_2$  are in  $\bar{B}_2' = C_2$  and are incident to  $c_0^2$ . Thus,  $d_{\bar{C}_2}(c_0^2) > 1$ . Now, let us prove that  $c_0^3 \notin V(C_2) = V(B_2')$ . Since  $B_2 \neq B_1$ , by item (iii) of Definition 3.1 (applied to  $S$  with  $i = 2$ ), we have  $b_0^3 \notin V(B_2)$ . Since  $b_0^1 \in V(B_2)$ , we know that  $V(B_2') \subset V(B_2)$ . Therefore,  $b_0^3 \notin V(B_2')$ . By the construction of the elements  $C_i$ , we have  $c_0^3 = b_0^3$ , which implies that  $c_0^3 \notin V(B_2') = V(C_2)$ .

**Item (ii):** For  $i = 3, \dots, r - 1$ , the element  $C_i$  contains  $c_0^2$ , and  $c_1^i = c_{*}^{i-1}$ .

Fix  $i \in \{3, \dots, r\}$ . Since  $b_0^1 \in B_i$ , by the definition of  $C_i$  we have that  $c_0^2 = b_0^1 \in C_i$ .

We shall prove that  $c_1^i = c_{*}^{i-1}$ . (a) If  $C_i = B_i$  and  $C_{i-1} = B_{i-1}$ , then the result follows by the definition of  $C$  and the fact that item (ii) of Definition 3.1 holds for the sequence  $S$ . (b) Suppose  $C_i = B_2'^-$ . In this case,  $B_i = B_2^-$ , and thus  $b_1^i = b_{\ell-1}^2$ . Since  $C_i = B_2'^-$ , we have that  $c_1^i = b_{\ell-1}^2$ . Combining the equalities above, we conclude that  $c_1^i = b_1^i$ . (b1) If  $C_{i-1} = B_{i-1}$  then  $c_{*}^{i-1} = b_{*}^{i-1}$ . Thus,  $c_1^i = b_1^i = b_{*}^{i-1} = c_{*}^{i-1}$  (as the middle equality holds because item (ii) of Definition 3.1 holds for  $S$ ). (b2) If  $C_{i-1} \neq B_{i-1}$ , then  $C_{i-1} = B_1'^-$ . The last equality implies that  $c_{*}^{i-1} = b_{+}^1$  and  $B_{i-1} = B_1^-$ . From the last equality, we obtain that  $b_{*}^{i-1} = b_{+}^1$ . Combining the equalities, we get  $c_{*}^{i-1} = b_{+}^1 = b_{*}^{i-1} = b_1^i = c_1^i$ .

(c) Suppose  $C_i = B_1'^-$ . The proof for this case is analogous to the proof of case (b), interchanging the occurrences of index 2 and index 1. We write the proof for completeness. In this case,  $B_i = B_1^-$ , and thus  $b_1^i = b_{\ell-1}^1$ . Since  $C_i = B_1'^-$ , we have that  $c_1^i = b_{\ell-1}^1$ . Combining the equalities above, we conclude that  $c_1^i = b_1^i$ . (c1) If  $C_{i-1} = B_{i-1}$  then

$c_*^{i-1} = b_*^{i-1}$ . Thus,  $c_1^i = b_1^i = b_*^{i-1} = c_*^{i-1}$ . (c2) If  $C_{i-1} \neq B_{i-1}$ , then  $C_{i-1} = B_2'^-$ . The last equality implies that  $c_*^{i-1} = b_+^2$  and  $B_{i-1} = B_2^-$ . From the last equality, we obtain that  $b_*^{i-1} = b_+^2$ . Combining the equalities, we get  $c_*^{i-1} = b_+^2 = b_*^{i-1} = b_1^i = c_1^i$ .

(d) Suppose  $C_i = B_i$  and  $C_{i-1} \neq B_{i-1}$ . If  $i = 3$  then  $C_2 = B_2'$ , and in this case,  $c_1^3 = b_1^3 = b_*^2 = c_*^2$ . If  $i > 3$ , then  $C_{i-1} = B_2'^-$  or  $C_{i-1} = B_1'^-$ . In both cases, it follows that  $c_*^{i-1} = b_*^{i-1}$ . Then using the fact that  $b_1^i = b_*^{i-1}$  (definition of  $S$ ), it follows that  $c_1^i = b_1^i = b_*^{i-1} = c_*^{i-1}$ .

**Item (iii):** For  $i = 3, \dots, r-1$ , if  $\bar{C}_i \neq \bar{C}_h$  for every  $h < i$ , then  $c_0^{i+1} \notin V(C_i)$ .

Fix  $i \in \{3, \dots, r-1\}$  and suppose  $\bar{C}_i \neq \bar{C}_h$  for every  $2 \leq h < i$ . Note that  $\bar{C}_i \neq \bar{B}_2'$ .

First we consider the case where  $\bar{C}_i = \bar{B}_1'$  or, equivalently,  $C_i = B_1'^-$ . Thus,  $B_i = B_1^-$ , and by item (iv) of Definition 3.1 applied to  $S$ , we have  $b_0^{i+1} \notin (V(\bar{B}_i) \cup \{b_0^2\}) = V(B_1') = V(C_i)$ . Since  $c_0^i = b_0^i$  for every  $i \geq 3$ , we have  $c_0^{i+1} \notin V(C_i)$ .

Now suppose that  $\bar{C}_i \neq \bar{B}_1'$ . Then,  $\bar{B}_i \neq \bar{B}_1$ . But we know that  $\bar{C}_i \neq \bar{C}_h$  for every  $h < i$ , which implies that  $B_i \neq B_h$  for every  $h < i$ . From item (iii) of Definition 3.1 (applied to  $S$ ), we have that  $b_0^{i+1} \notin V(B_i)$ . Since  $c_0^{i+1} = b_0^{i+1}$ , we have  $c_0^{i+1} \notin V(B_i)$ . Since  $\bar{C}_i \neq \bar{B}_1', \bar{B}_2'$ , we conclude that  $C_i = B_i$ , and therefore,  $c_0^{i+1} \notin V(C_i)$ .

**Item (iv):** for  $i = 3, \dots, r-1$ , if  $\bar{C}_i = \bar{C}_2$  then  $c_0^{i+1} \notin V(C_i) \cup \{c_0^3\}$ .

Fix  $i \in \{3, \dots, r-1\}$  and suppose  $\bar{C}_i = \bar{C}_2$ . In this case,  $C_i = B_2'^-$ . We shall prove that  $c_0^{i+1} \notin V(B_2') \cup \{c_0^3\}$ . Note that, by the definition of  $C_i$ , we have  $B_i = B_2^-$ . Thus, by item (v) of Definition 3.1 applied to  $S$  with parameters  $i$  and  $h = 2$ , we have

$$b_0^{i+1} \notin V(\bar{B}_2 - f_2 + e_1 - e_2 + f_3). \quad (1)$$

Note that  $\bar{C}_2 = \bar{B}_2' = \bar{B}_2 - f_2 + e_1$ . Therefore,

$$V(\bar{B}_2 - f_2 + e_1 - e_2 + f_3) = V(\bar{C}_2 - e_2 + f_3) = V(\bar{C}_2) \cup \{b_0^3\}. \quad (2)$$

Recall that  $c_0^{i+1} = b_0^{i+1}$  for every  $i \geq 2$ . Then, by (1) and (2), we have  $c_0^{i+1} \notin V(C_2) \cup \{c_0^3\}$ .

**Item (v):** for  $i = 3, \dots, r-1$ , if  $\bar{C}_i = \bar{C}_h$  for some  $2 < h < i$ , then  $c_0^{i+1} \notin V(\bar{C}_i - f_h^* + e_{h-1}^* - e_h^* + f_{h+1}^*)$ .

Fix  $i \in \{3, \dots, r-1\}$  and suppose that  $\bar{C}_i = \bar{C}_h$  for some  $2 < h < i$ . Note that we have  $C_i = C_h^-$ , and thus  $\bar{B}_i = \bar{B}_h$  and, by Corollary 3.3,  $\bar{C}_i \neq \bar{B}_1', \bar{B}_2'$ . By item (v) of Definition 3.1 applied to  $S$ , we have

$$b_0^{i+1} \notin V(\bar{B}_h - f_h + e_{h-1} - e_h + f_{h+1}). \quad (3)$$

Recall that, since  $i \geq 3$ , we have  $c_0^{i+1} = b_0^{i+1}$ ,  $f_i^* = f_i$  and  $e_i^* = e_i$ . Therefore, from (3), we have

$$c_0^{i+1} \notin V(\bar{C}_h - f_h^* + e_{h-1}^* - e_h^* + f_{h+1}^*).$$

We concluded the proof that  $S'$  is an augmenting sequence of  $\mathcal{B}''$ . But, since  $S$  is a full-augmenting sequence of  $\mathcal{B}$ , we know that  $b_0^1 \notin V(B_r)$  (then, clearly  $\bar{B}_r \neq \bar{B}_1, \bar{B}_2$ ). But since  $c_0^2 = b_0^1$ , we conclude that  $S'$  is a full-augmenting sequence of  $\mathcal{B}''$ .

Since  $|S'| = r - 1$ , by the induction hypothesis,  $G$  admits a  $k$ -complete  $\ell$ -tracking decomposition  $\mathcal{B}'$  such that  $\tau(\mathcal{B}') < \tau(\mathcal{B}'') \leq \tau(\mathcal{B})$  and  $\mathcal{B}'(v) = \mathcal{B}''(v) = \mathcal{B}(v)$  for every vertex  $v$  of  $G$ .  $\square$

The following concept and lemma are important in the construction of full-augmenting sequences.

**Definition 3.7.** Let  $\ell$  be a positive integer. Let  $G$  be a graph and  $\mathcal{B}$  be an  $\ell$ -tracking decomposition of  $G$ . We say that  $\mathcal{B}$  is feasible if for every  $v \in V(G)$  the following holds: if  $T$  is a vanilla  $\ell$ -trail of  $G$  (not necessarily in  $\mathcal{B}$ ) that contains  $v$  as an internal vertex, then there exists a hanging edge  $vw$  at  $v$  in  $\mathcal{B}$  such that  $w \notin V(T)$ .

**Lemma 3.8.** Let  $\ell$  and  $k$  be a positive integers and  $G$  be a bipartite graph. If  $k \geq \lceil (\ell + 1)/2 \rceil$  and  $\mathcal{B}$  is an  $k$ -complete  $\ell$ -tracking decomposition of  $G$ , then  $\mathcal{B}$  is feasible.

*Proof.* Let  $\ell, k, G$  and  $\mathcal{B}$  be as in the hypothesis of the lemma. Fix  $v \in V(G)$  and suppose  $T$  is a vanilla  $\ell$ -trail of  $G$  that contains  $v$ . Since  $\mathcal{B}$  is  $k$ -complete,  $\text{hang}(v, \mathcal{B}) > k$ . Let  $vw_1, \dots, vw_{k+1}$  be hanging edges at  $v$  in  $\mathcal{B}$ .

We claim that there exists an index  $1 \leq i \leq k + 1$  such that  $w_i \notin V(T)$ . Let  $W = \{w_1, \dots, w_{k+1}\}$ . Let  $G = (A, B; E)$  and suppose, without loss of generality, that  $v \in A$ . Since  $G$  is bipartite,  $W \subset B$ . Furthermore, since  $T$  contains at most  $\ell + 1$  vertices,  $|V(T) \cap B| \leq \lceil (\ell + 1)/2 \rceil \leq k$ . But since  $|W| = k + 1$ , we conclude that there exists a vertex  $w \in W$  such that  $w \notin V(T)$ .  $\square$

Recall that, for a tracking  $B$ , we denote by  $\tau(B)$  the number of end-vertices of  $B$  that have degree greater than 1, and for a tracking decomposition  $\mathcal{B}$ , we denote by  $\tau(\mathcal{B})$  the sum  $\sum_{B \in \mathcal{B}} \tau(B)$ .

**Lemma 3.9.** Let  $\ell$  be a positive integer,  $G$  be a graph and  $\mathcal{B}$  be an  $\ell$ -tracking decomposition of  $G$ . If  $\mathcal{B}$  is feasible and  $\tau(\mathcal{B}) > 0$ , then  $\mathcal{B}$  contains a full-augmenting sequence.

*Proof.* Let  $\ell, G$  and  $\mathcal{B}$  be as in the hypothesis of the lemma. First, let us show that  $\mathcal{B}$  contains an augmenting sequence. Since  $\tau(\mathcal{B}) > 0$ , the tracking decomposition  $\mathcal{B}$  contains a tracking  $B_1$  that does not induce a path. Let  $B_1 = b_0^1 b_1^1 \dots b_\ell^1$ , where  $d_{\bar{B}_1}(b_0^1) > 1$ .

Since  $\mathcal{B}$  is feasible, there is a hanging edge  $b_*^1 w$  at (the internal vertex)  $b_*^1$  in  $\mathcal{B}$  such that  $w \notin V(B_1)$ . Let  $B_2$  be the element of  $\mathcal{B}$  that contains the edge  $b_*^1 w$ . Then, it is easy to verify that  $B_1 B_2$  is an augmenting sequence of  $\mathcal{B}$ .



Let  $S = B_1 B_2 \cdots B_r$  be a maximal augmenting sequence of  $\mathcal{B}$ . Suppose by contradiction that  $S$  is not a full-augmenting sequence, i.e.,  $b_0^1 \in V(B_r)$ .

Now we show how to obtain an element  $B_{r+1}$  of  $\mathcal{B}$  such that  $S' = B_1 \cdots B_r B_{r+1}$  is an augmenting sequence, contradicting the maximality of  $S$ . Since  $S$  is an augmenting sequence, item (i1) of Definition 3.1 holds, and items (ii)–(v) of Definition 3.1 hold for  $i = 1, \dots, r-1$ . Since  $S$  is not a full-augmenting sequence,  $B_r$  contains  $b_0^1$ . Our aim is to find an element  $B_{r+1}$  for which items (iii)–(v) of Definition 3.1 hold for  $i = r$ . Before continuing, note that  $b_*^i$  is a vertex of  $B_i$  in the tracking  $P_i = b_1^i b_2^i \cdots b_*^i$ , and therefore  $b_*^i$  is always an internal vertex of  $B_i$  (because  $B_i$  contains the tracking  $P_i b_0^1$ ). Now, note that exactly one of the following holds: (a)  $\bar{B}_r \neq \bar{B}_h$ , for every  $h < r$ ; (b)  $\bar{B}_r = \bar{B}_1$ ; or (c)  $\bar{B}_r = \bar{B}_h$  for some  $1 < h < r$ .

- (a) In this case, by the feasibility of  $\mathcal{B}$ , considering  $T = \bar{B}_r$  and  $v = b_*^r$ , there exists a hanging edge  $b_*^r z$  at  $b_*^r$  such that  $z \notin V(B_r)$ . Let  $B_{r+1}$  be the element of  $\mathcal{B}$  containing  $b_*^r z$ . We can suppose without loss of generality that  $z = b_0^{r+1}$  (otherwise,  $z = b_\ell^{r+1}$  and we choose  $B_{r+1}^-$  instead of  $B_{r+1}$ ). Then,  $b_1^{r+1} = b_*^r$ ,  $b_0^{r+1} = z \notin V(B_r)$ , and item (iii) of Definition 3.1 holds for  $i = r$ .
- (b) In this case, we have  $B_r = B_1^- = b_\ell^1 P_r b_0^1 P_1^- b_0^1$ . By the feasibility of  $\mathcal{B}$ , considering  $v = b_*^r$  and  $T = \bar{B}_1 - e_1 + f_2$  (note that  $T$  is induced by the tracking  $b_\ell^1 P_r b_0^1 P_1 b_0^2$ ), there exists a hanging edge  $b_*^r z$  at  $b_*^r$  such that  $z \notin V(T)$ . Note that  $V(T) = V(B_1) \cup \{b_0^2\}$ . Let  $B_{r+1}$  be the element of  $\mathcal{B}$  containing  $b_*^r z$ . As in the previous case, we may assume that  $z = b_0^{r+1} \notin V(B_r) \cup \{b_0^2\}$ . Then,  $b_1^{r+1} = b_*^r$  and item (iv) of Definition 3.1 holds for  $i = r$ .
- (c) In this case, we have  $B_r = b_\ell^h P_r b_0^1 P_h^- b_0^h$ . Since  $S$  is an augmenting sequence, by item (ii) of Definition 3.1,  $b_1^h = b_*^{h-1}$  and  $b_1^{h+1} = b_*^h$ . Then, since  $b_{s(h-1)}^{h-1} = b_{s(h)}^h = b_0^1$ , we conclude that  $e_{h-1} = b_1^h b_0^1$  and  $f_{h+1} = b_*^h b_0^{h+1}$  are edges of  $B_{h-1}$  and  $B_{h+1}$ , respectively. Put  $T = \bar{B}_r - b_1^h b_0^1 + b_1^h b_0^1 - b_*^h b_0^1 + b_*^h b_0^{h+1} = \bar{B}_r - f_h + e_{h-1} - e_h + f_{h+1}$  (note that  $T$  is induced by the tracking  $b_\ell^h P_r b_0^1 P_h b_0^{h+1}$ ). Since  $T$  is a vanilla  $\ell$ -trail, by the feasibility of  $\mathcal{B}$ , considering  $v = b_*^r$  and  $T$ , there is a hanging edge  $b_*^r z$  at  $b_*^r$  such that  $z \notin V(T)$ . Let  $B_{r+1}$  be the element of  $\mathcal{B}$  containing  $b_*^r z$ . As in the previous case, we may assume that  $z = b_0^{r+1} \notin V(T)$ . Then,  $b_1^{r+1} = b_*^r$  and item (v) of Definition 3.1 holds for  $i = r$ .

We just proved that there exists an element  $B_{r+1}$  of  $\mathcal{B}$  such that  $S' = B_1 B_2 \cdots B_r B_{r+1}$  is an augmenting sequence of  $\mathcal{B}$ , a contradiction to the maximality of  $S$ . Therefore,  $S$  is a full-augmenting sequence.  $\square$

The next result follows directly from Lemmas 3.4 and 3.9.

**Corollary 3.10.** *Let  $\ell$  be a positive integer and let  $G$  be a graph. If  $\mathcal{B}$  is a feasible  $k$ -complete  $\ell$ -tracking decomposition of  $G$  and  $\tau(\mathcal{B}) > 0$ , then there is an  $\ell$ -tracking decomposition  $\mathcal{B}'$  of  $G$  such that the following holds.*

- $\tau(\mathcal{B}') < \tau(\mathcal{B})$ ;
- $\mathcal{B}'(v) = \mathcal{B}(v)$  for every  $v \in V(G)$ ;
- $\mathcal{B}'$  is  $k$ -complete.

The next lemma, the main result of this section, combines Lemma 3.8 and Corollary 3.10 to obtain  $\ell$ -path tracking decompositions from  $\lceil(\ell + 1)/2\rceil$ -complete  $\ell$ -tracking decompositions.

**Lemma 3.11** (The Disentangling Lemma). *Let  $\ell$  and  $k$  be positive integers and let  $G$  be a bipartite graph. If  $k \geq \lceil(\ell + 1)/2\rceil$  and  $\mathcal{B}$  is a  $k$ -complete  $\ell$ -tracking decomposition of  $G$ , then  $G$  admits a  $k$ -complete  $\ell$ -path tracking decomposition  $\mathcal{B}'$  such that  $\mathcal{B}'(v) = \mathcal{B}(v)$  for every vertex  $v$  of  $G$ .*

*Proof.* Let  $\ell$ ,  $k$ ,  $G$  and  $\mathcal{B}$  be as in the hypothesis of the lemma. Let  $\mathbb{B}$  be the set of all  $k$ -complete  $\ell$ -tracking decompositions  $\mathcal{B}'$  of  $G$  such that  $\mathcal{B}'(v) = \mathcal{B}(v)$  for every vertex  $v$  of  $G$ . By the hypothesis,  $\mathbb{B} \neq \emptyset$ . Let  $\tau^* = \min\{\tau(\mathcal{B}'): \mathcal{B}' \in \mathbb{B}\}$  and let  $\mathcal{B}_{\min}$  be an element of  $\mathbb{B}$  such that  $\tau(\mathcal{B}_{\min}) = \tau^*$ . If  $\tau^* = 0$ , then  $\mathcal{B}_{\min}$  is an  $\ell$ -path tracking decomposition and the proof is complete. Then, assume  $\tau^* > 0$ . By Lemma 3.8,  $\mathcal{B}_{\min}$  is a feasible  $\ell$ -tracking decomposition. Since  $\tau(\mathcal{B}_{\min}) > 0$ , by Corollary 3.10 (applied with  $k$ ,  $\ell$ ,  $G$  and  $\mathcal{B}_{\min}$ ), there exists an  $k$ -complete  $\ell$ -tracking decomposition  $\mathcal{B}'$  of  $G$  such that  $\tau(\mathcal{B}') < \tau(\mathcal{B}_{\min}) = \tau^*$  and  $\mathcal{B}'(v) = \mathcal{B}(v)$  for every vertex  $v$  of  $G$ . Therefore,  $\mathcal{B}'$  is an element of  $\mathbb{B}$  with  $\tau(\mathcal{B}') < \tau(\mathcal{B}_{\min})$ , a contradiction to the minimality of  $\tau^*$ .  $\square$

#### 4. FACTORIZATIONS

The goal of this section is to show that some bipartite highly edge-connected graphs admit “well structured” decompositions, called *bifactorizations*, which are important structures in the proof of the main theorems of this paper (shown in Section 7). The diagram of Figure 1 shows how the results of this section are related.

**4.1. Fractional factorizations.** We extend ideas developed in [6] in order to prove that some highly edge-connected bipartite graphs admit structured factorizations. Let us start with some definitions.

**Definition 4.1** (Factor). *Let  $r$  and  $k$  be positive integers and  $G = (V, E)$  be a graph. Let  $X \subset V$  and  $F \subset E$ . We say that  $F$  is an  $(X, r, k)$ -factor of  $G$  if, for every  $v \in X$ , we have  $d_F(v) = (r/k)d_G(v)$ .*

**Definition 4.2** (Fractional factorization). *Let  $k$  and  $\ell$  be positive integers such that  $k - \ell$  is a positive even number. Let  $G = (V, E)$  be a graph and let  $X \subset V$ . We say that a partition  $\mathcal{F} = \{M_1, \dots, M_\ell, F_1, \dots, F_{(k-\ell)/2}\}$  of  $E$  is an  $(X, \ell, k)$ -fractional factorization of  $G$  if the following holds.*

- $M_i$  is an  $(X, 1, k)$ -factor of  $G$ , for  $1 \leq i \leq \ell$ ;
- $F_j$  is an Eulerian  $(X, 2, k)$ -factor of  $G$ , for  $1 \leq j \leq (k - \ell)/2$ .

Note that, if  $G$  contains an  $(X, 1, k)$ -factor, then  $d_G(v)$  is divisible by  $k$  for every  $v \in X$ . Therefore, this fact implies that, if  $G$  admits an  $(X, \ell, k)$ -fractional factorization, then  $d(v)$  is divisible by  $k$  for every  $v \in X$ . The next lemma is the core of this section.

**Lemma 4.3.** *Let  $k$  be a positive integer. If  $G = (A, B; E)$  is a  $2k$ -edge-connected bipartite graph such that  $d_G(v)$  is divisible by  $2k+1$  for every  $v \in A$ , then  $G$  admits an  $(A, 1, 2k+1)$ -fractional factorization.*

*Proof.* Let  $G = (A, B; E)$  be as in the hypothesis. First, we want to apply Lemma 2.1 to obtain a  $2k$ -edge-connected graph  $G'$  with maximum degree  $4k - 1$ . To do this, for every vertex  $v \in B$ , we take integers  $s_v \geq 1$  and  $0 \leq r_v < 2k$  such that  $d_G(v) = 2ks_v + r_v$ . We put  $d_1^v = 2k + r_v$  and  $d_2^v = \dots = d_{s_v}^v = 2k$ . Furthermore, for every vertex  $v \in A$ , we put  $s_v = d_G(v)/(2k + 1)$  and  $d_i^v = 2k + 1$  for  $1 \leq i \leq s_v$ . By Lemma 2.1 (applied with parameters  $2k$  and the integers  $s_v, d_i^v$  ( $1 \leq i \leq s_v$ ) for every  $v \in V(G)$ ), there exists a  $2k$ -edge-connected bipartite graph  $G'$  obtained from  $G$  by splitting each vertex  $v$  of  $A$  into  $s_v$  vertices of degree  $2k + 1$ , and each vertex  $v$  of  $B$  into a vertex of degree  $2k + r_v < 4k$  and  $s_v - 1$  vertices of degree  $2k$ . Let  $A'$  and  $B'$  be the set of vertices of  $G'$  obtained from the vertices of  $A$  and  $B$ , respectively. For ease of notation, if  $v \in (A' \cup B') \setminus (A \cup B)$  we also denote by  $v$  the vertex in  $A \cup B$  that gave rise to  $v$ .

The next step is to obtain a  $(2k + 1)$ -regular multigraph  $G^*$  from  $G'$  by using lifting operations. For this, we will add some edges to  $A'$  and remove the even-degree vertices of  $B'$  by successive applications of Mader's Lifting Theorem as follows. Let  $G'_0, G'_1, \dots, G'_\lambda$  be a maximal sequence of graphs such that  $G'_0 = G'$  and (for  $i \geq 0$ )  $G'_{i+1}$  is the graph obtained from  $G'_i$  by the application of an admissible lifting at an arbitrary vertex  $v$  with  $d_{G'_i}(v) \notin \{1, 2, 2k + 1\}$ .

Recall that, given any two distinct vertices of  $G'$ , say  $x$  and  $y$ , we denote by  $p_{G'}(x, y)$  the maximum number of pairwise edge-disjoint paths joining  $x$  and  $y$  in  $G'$ . We claim that  $p_{G'_i}(x, y) \geq 2k$  for any  $x, y$  in  $A'$  and every  $i \geq 0$ . Clearly,  $p_{G'_0}(x, y) \geq 2k$  holds for any  $x, y$  in  $A'$ , since  $G'$  is  $2k$ -edge-connected. Fix  $i \geq 0$  and suppose  $p_{G'_i}(x, y) \geq 2k$  holds for any  $x, y$  in  $A'$ . Let  $x, y$  be two vertices in  $A'$ . Since  $G'_{i+1}$  is a graph obtained from  $G'_i$  by the application of an admissible lifting at a vertex  $v$  in  $B'$ , we have  $p_{G'_{i+1}}(x, y) \geq p_{G'_i}(x, y) \geq 2k$ .

We claim that, if  $v \in B'$  then  $d_{G'_\lambda}(v) \in \{2, 2k + 1\}$ . Suppose, for a contradiction, that there is a vertex  $v$  in  $B'$  such that  $d_{G'_\lambda}(v) \notin \{2, 2k + 1\}$ . Note that  $d_{G'_i}(u) \geq d_{G'_{i+1}}(u) \geq 2$  for every  $u \in V(G')$  and every  $0 \leq i \leq \lambda$ . Since  $d_{G'}(u) \leq 4k - 1$  for every  $u \in V(G')$ , we have  $2 \leq d_{G'_i}(u) \leq 4k - 1$  for every  $0 \leq i \leq \lambda$ . Therefore,  $2 \leq d_{G'_\lambda}(v) \leq 4k - 1$ . Since  $d_{G'_\lambda}(v) \leq 4k - 1$ , and for any two neighbors  $x$  and  $y$  of  $v$  we have  $p_{G'_\lambda}(x, y) \geq 2k$ , Lemma 2.3 implies that  $v$  is not a cut-vertex of  $G'_\lambda$ . Then, by Mader's Lifting Theorem (Theorem 2.2) applied to  $G'_\lambda$ , there is an admissible lifting at  $v$ . Therefore,  $G'_0, G'_1, \dots, G'_\lambda$  is not maximal, a contradiction.

In  $G'_\lambda$  the set  $B'$  may have some vertices of degree 2. For every such vertex  $v$ , if  $u$  and  $w$  are the neighbours of  $v$ , we apply a  $uw$ -lifting at  $v$ , and remove the vertex  $v$ , i.e., we perform a suppression of  $v$ . Let  $G^*$  be the graph obtained by applying this process to all vertices of degree 2 in  $B'$ . Note that the number of pairwise edge-disjoint paths joining two distinct vertices of  $A'$  remains the same, i.e.,  $p_{G^*}(x, y) \geq p_{G'_\lambda}(x, y) \geq 2k$  for every  $x, y$  in  $A'$ . Clearly, the set of vertices of  $G^*$  that belong to  $B'$  is an independent set; we denote it by  $B^*$  (eventually,  $B^* = \emptyset$ ). Furthermore, every vertex in  $B^*$  has degree  $2k + 1$ .

**Claim 4.4.**  $G^*$  is  $2k$ -edge connected.

*Proof.* Let  $Y \subset V(G^*)$ . Suppose there is at least one vertex  $x$  of  $A'$  in  $Y$  and at least one vertex  $y$  of  $A'$  in  $V(G^*) - Y$ . Since there are at least  $2k$  edge-disjoint paths joining  $x$  to  $y$ , there are at least  $2k$  edges with vertices in both  $Y$  and  $V(G^*) - Y$ . Now, suppose that  $A' \subset Y$  (otherwise  $A' \subset V(G^*) - Y$  and we take  $V(G^*) - Y$  instead of  $Y$ ), and then  $V(G^*) - Y \subset B^*$ . Since  $B^*$  is an independent set, all edges with a vertex in  $V(G^*) - Y$  must have the other vertex in  $A'$ . Since every vertex in  $B^*$  has degree  $2k + 1$ , there are at least  $2k + 1$  edges with vertices in both  $Y$  and  $V(G^*) - Y$ .  $\square$

We conclude that  $G^*$  is a  $2k$ -edge-connected  $(2k + 1)$ -regular multigraph with vertex-set  $A' \cup B^*$ , where  $B^*$  is an independent set.

Since every vertex of  $G^*$  has odd degree,  $|V(G^*)|$  is even. Thus,  $G^*$  is a  $2k$ -edge-connected  $(2k + 1)$ -regular multigraph of even order. By Theorem 2.6,  $G^*$  contains a perfect matching  $M^*$ . Since the multigraph  $J^* = G^* - M^*$  is  $2k$ -regular, Theorem 2.7 implies that  $J^*$  admits a decomposition into 2-factors, say  $F_1^*, \dots, F_k^*$ . Therefore,  $M^*, F_1^*, \dots, F_k^*$  is a partition of  $E(G^*)$ .

Now, let us get back to the bipartite graph  $G$ . Let  $xy$  be an edge of  $G^*$ . If  $x \in A'$  and  $y \in B^*$ , then  $xy$  corresponds to an edge of  $G$ . On the other hand, if  $x, y \in A'$ , then there is a vertex  $v_{xy}$  of  $B'$  and two edges  $xv_{xy}$  and  $v_{xy}y$  in  $E(G')$ . Furthermore,  $xy$  was obtained by an  $xy$ -lifting at  $v_{xy}$  (either by an application of Mader's Lifting Theorem or by the suppression of vertices of degree 2). Then, each edge of  $G^*$  represents an edge of  $G$  or a 2-path in  $G$  such that the internal vertices of these 2-paths are always in  $B$ . For every edge  $xy \in E(G^*)$ , define  $f(xy) = \{xy\}$  if  $x \in A'$  and  $y \in B^*$ , and  $f(xy) = \{xv_{xy}, v_{xy}y\}$  if  $x, y \in A'$ . Note that  $f(xy) \subset E(G)$  for every edge  $xy \in E(G^*)$ . For a set  $S \subset E(G^*)$ , put  $f(S) = \cup_{e \in S} f(e)$ . The partition of  $E(G^*)$  into  $M^*, F_1^*, \dots, F_k^*$  induces a partition of  $E(G)$  into  $M = f(M^*)$  and  $F_i = f(F_i^*)$  for  $1 \leq i \leq k$ .

We will prove that  $\{M, F_1, \dots, F_k\}$  is an  $(A, 1, 2k + 1)$ -fractional factorization. Fix  $i \in \{1, \dots, k\}$ . We will show that  $M$  is an  $(A, 1, 2k + 1)$ -factor of  $G$  and  $F_i$  is an Eulerian  $(A, 2, 2k + 1)$ -factor of  $G$ . Let  $v$  be a vertex of  $A$  in  $G$  and put  $d'(v) = d(v)/(2k + 1)$ . Then, we know that  $v$  is represented by  $d'(v)$  vertices in  $G^*$ . Since  $M^*$  is a perfect matching in  $G^*$ , there are  $d'(v)$  edges of  $M$  entering  $v$  and, since  $F_i^*$  is a 2-factor in  $G^*$ , there are  $2d'(v)$  edges of  $F_i$  incident to  $v$ . Finally, since  $F_i^*$  is Eulerian, the set  $F_i$  is Eulerian, concluding the proof.  $\square$

**Corollary 4.5.** *Let  $k$  be a positive integer. If  $G = (A, B; E)$  is a  $32k$ -edge-connected bipartite graph such that  $d_G(v)$  is divisible by  $2k + 2$  for every  $v \in A$ , then  $G$  admits an  $(A, 2, 2k + 2)$ -fractional factorization.*

*Proof.* Let  $k$  and  $G = (A, B; E)$  be as in the hypothesis. We claim that  $G$  contains an  $(A, 1, 2k + 2)$ -factor  $F$  such that  $G - F$  is  $2k$ -edge-connected (note that this implies that  $d_{G-F}(v)$  is divisible by  $2k + 1$  for every  $v \in A$ ).

Since  $G$  is  $32k = 16k \lceil (2k + 2)/(2k + 1) \rceil$ -edge-connected, by Lemma 2.10 (applied with parameters  $2k + 1$ ,  $m = 2k$  and  $r = 1$ ), the graph  $G$  admits a decomposition into graphs  $G_k$  and  $G_r$  such that  $G_k$  is  $2k$ -edge-connected and  $d_{G_k}(v) = ((2k + 1)/(2k + 2))d_G(v)$ , and  $d_{G_r}(v) = (1/(2k + 2))d_G(v)$  for every  $v \in A$ . Therefore,  $E(G_r)$  is an  $(A, 1, 2k + 2)$ -factor.

By Lemma 4.3,  $G_k$  admits an  $(A, 1, 2k + 1)$ -fractional factorization  $\mathcal{F}$ . Therefore, since  $d_{G_k}(v) = ((2k + 1)/(2k + 2))d_G(v)$  for every  $v \in A$ , we conclude that  $\mathcal{F} + E(G_r)$  is an  $(A, 2, 2k + 2)$ -fractional factorization of  $G$ . □

**4.2. Bifactorizations.** To obtain a decomposition of highly edge-connected bipartite graphs  $G$  into paths of a fixed length  $\ell$ , we will combine fractional factorizations to obtain first an  $\ell$ -tracking decomposition. More specifically, we decompose  $G$  into graphs  $G_1$  and  $G_2$  and then we combine a fractional factorization of  $G_1$  with a fractional factorization of  $G_2$ . This process, called bifactorizations, is defined as follows.

**Definition 4.6** (Bifactorization). *Let  $k$  and  $\ell$  be positive integers such that  $k - \ell$  is a positive even number, and let  $G = (A_1, A_2; E)$  be a bipartite graph.*

*Let  $\mathcal{F}_1, \mathcal{F}_2$  be families of subsets of  $E$  and put  $G_i = G[\cup_{F \in \mathcal{F}_i} F]$ , for  $i = 1, 2$ . We say that  $\mathbb{F} = (\mathcal{F}_1, \mathcal{F}_2)$  is an  $(\ell, k)$ -bifactorization of  $G$  if the following holds.*

- (i)  $\{G_1, G_2\}$  is a decomposition of  $G$  ;
- (ii)  $\mathcal{F}_i$  is an  $(A_i, \ell, k)$ -fractional factorization of  $G_i$ , for  $1 \leq i \leq 2$ .

*If  $G$  admits an  $(\ell, k)$ -bifactorization, we say that  $G$  is  $(\ell, k)$ -bifactorable.*

The next concept will be used to guarantee that  $G_1$  and  $G_2$  have a sufficiently large minimum degree.

**Definition 4.7** (Strong bifactorization). *Let  $k$  and  $\ell$  be positive integers. Let  $G = (A_1, A_2; E)$  be a bipartite graph that admits an  $(\ell, k)$ -bifactorization  $\mathbb{F} = (\mathcal{F}_1, \mathcal{F}_2)$ . Let  $E_i = \cup_{F \in \mathcal{F}_i} F$  for  $1 \leq i \leq 2$ . Let  $p = 1$  if  $\ell$  is odd, and  $p = 2$  if  $\ell$  is even. We say that  $\mathbb{F}$  is strong if  $d_{E_i}(v) \geq (k/p)((k/p) + p)$  for every  $v$  in  $A_i$  for  $1 \leq i \leq 2$ . If  $G$  admits a strong  $(\ell, k)$ -bifactorization, we say that  $G$  is strongly  $(\ell, k)$ -bifactorable.*

For ease of notation, if  $F$  belongs to either  $\mathcal{F}_1$  or  $\mathcal{F}_2$ , then we say that  $F$  is an element of  $\mathbb{F}$ . In what follows, we give sufficient conditions for a bipartite graph to be strongly bifactorable.

**Lemma 4.8.** *Let  $k$  be a positive integer. Let  $r = (2k+1)(2k+2)$ . If  $G$  is a  $2(6k+2r+1)$ -edge-connected bipartite graph such that  $|E(G)|$  is divisible by  $2k+1$ , then  $G$  is strongly  $(1, 2k+1)$ -bifactorable.*

*Proof.* Let  $k$ ,  $r$  and  $G = (A, B; E)$  be as in the hypothesis. By Lemma 2.5 (applied with  $2k+1$  and  $r$ ), the graph  $G$  can be decomposed into two spanning edge-disjoint  $r$ -edge-connected graphs  $G_1$  and  $G_2$  such that all vertices of  $A$  have degree divisible by  $2k+1$  in  $G_1$ , and all vertices of  $B$  have degree divisible by  $2k+1$  in  $G_2$ . But since  $r \geq 2k$ , by Lemma 4.3 (applied with  $k$ ), we conclude that  $G_1$  admits an  $(A, 1, 2k+1)$ -fractional factorization and  $G_2$  admits a  $(B, 1, 2k+1)$ -fractional factorization. Therefore,  $G$  is  $(1, 2k+1)$ -bifactorable. Since  $G_1$  and  $G_2$  are  $r$ -edge-connected, we have  $d_{G_1}(v) \geq (2k+1)(2k+2)$  for every  $v \in A$ , and  $d_{G_2}(v) \geq (2k+1)(2k+2)$  for every  $v \in B$ , from where we conclude that  $G$  is strongly  $(1, 2k+1)$ -bifactorable.  $\square$

The proof of the next lemma can be easily obtained by replacing Lemma 4.3 with Corollary 4.5 in the proof of Lemma 4.8.

**Lemma 4.9.** *Let  $k$  be a positive integer. Let  $r = \max\{32k, (k+1)(k+3)\}$ . If  $G$  is a  $2(6k+2r+4)$ -edge-connected bipartite graph such that  $|E(G)|$  is divisible by  $2k+2$ , then  $G$  is strongly  $(2, 2k+2)$ -bifactorable.*

## 5. DECOMPOSITION INTO PATHS OF ODD LENGTH

We present now a definition which is central to what follows. Before that, we recall that given an  $\ell$ -tracking decomposition  $\mathcal{B}$  of a graph  $G$ ,  $\mathcal{B}(v)$  denotes the number of edges of  $G$  incident to  $v$  that are starting edges of trackings in  $\mathcal{B}$  that start at  $v$ , or ending edges of trackings in  $\mathcal{B}$  that end at  $v$ .

**Definition 5.1** (Balanced tracking decomposition – odd case). *Let  $k$  be a positive integer. Let  $G = (A, B; E)$  be a bipartite graph that admits a  $(1, 2k+1)$ -bifactorization  $\mathbb{F} = (\mathcal{F}_1, \mathcal{F}_2)$ , and let  $G_i = G[\bigcup_{F \in \mathcal{F}_i} F]$  for  $i = 1, 2$ . Let  $M_1$  be the  $(A, 1, 2k+1)$ -factor of  $\mathbb{F}$  and  $M_2$  be the  $(B, 1, 2k+1)$ -factor of  $\mathbb{F}$ . We say that a  $(2k+1)$ -tracking decomposition  $\mathcal{B}$  of  $G$  is  $\mathbb{F}$ -balanced if the following holds.*

- $\mathcal{B}(v) = d_{G_1}(v)/(2k+1) + d_{M_2}(v)$  for every  $v \in A$ ;
- $\mathcal{B}(v) = d_{G_2}(v)/(2k+1) + d_{M_1}(v)$  for every  $v \in B$ .

Our aim in this section is to prove Theorem 5.5, which states that one may obtain an  $\mathbb{F}$ -balanced  $(k+1)$ -complete path tracking decomposition from a strong  $(1, 2k+1)$ -bifactorization  $\mathbb{F}$ . The proof of Theorem 5.5 is by induction on  $k$ . The base of the induction is precisely the statement of the next lemma, whose proof can be seen in Thomassen [21] (we present it for completeness).

**Lemma 5.2.** *Let  $G$  be a bipartite graph that admits a  $(1, 3)$ -bifactorization  $\mathbb{F}$ . Then  $G$  admits an  $\mathbb{F}$ -balanced 3-path tracking decomposition.*

*Proof.* Let  $G = (A, B; E)$  be a bipartite graph that admits a  $(1, 3)$ -bifactorization  $\mathbb{F} = (\mathcal{F}_1, \mathcal{F}_2)$  and put  $\mathcal{F}_i = \{M_i, F_i\}$  for  $i = 1, 2$ . Let  $\mathcal{C}_1$  be the set of components of  $G[F_1]$ . Let  $T$  be an element of  $\mathcal{C}_1$  and  $B_T = a_0 b_0 a_1 b_1 \cdots a_s b_s a_0$  be a tracking of  $T$ , where  $a_i \in A$  and  $b_i \in B$ , for  $1 \leq i \leq s$ . We have that  $\mathcal{B}'_T = \{a_i b_i a_{i+1} : 0 \leq i \leq s\}$ , taking  $a_{s+1} = a_0$ , is a 2-tracking decomposition of  $T$  in which every tracking has its end-vertices in  $A$ . Therefore,  $\mathcal{B}'_1 = \cup_{T \in \mathcal{C}_1} \mathcal{B}'_T$  is a 2-tracking decomposition of  $G[F_1]$  in which every tracking has its end-vertices in  $A$ . Analogously,  $G[F_2]$  admits a 2-tracking decomposition  $\mathcal{B}'_2$  in which every tracking has its end-vertices in  $B$ .

Let  $G_i = G[M_i \cup F_i]$  for  $i = 1, 2$ . Note that, since  $M_1$  is an  $(A, 1, 3)$ -factor and  $F_1$  is an  $(A, 2, 3)$ -factor of  $G_1$ , we have  $d_{F_1}(v) = (2/3)d_{G_1}(v) = 2d_{M_1}(v)$ , for every vertex  $v$  in  $A$ . Note that the number of trackings in  $\mathcal{B}'_1$  that end at a vertex  $v$  equals  $\frac{1}{2}d_{F_1}(v) = d_{M_1}(v)$ . Thus, we can extend each tracking  $B$  of  $\mathcal{B}'_1$  by adding an edge of  $M_1$  to the start vertex of  $B$ , obtaining a 3-path tracking decomposition  $\mathcal{B}_1$  of  $G_1$ . Analogously, we can extend each tracking  $T$  of  $\mathcal{B}'_2$  by adding an edge of  $M_2$  to the starting vertex of  $T$ , obtaining a 3-path tracking decomposition  $\mathcal{B}_2$  of  $G_2$ .

Put  $\mathcal{B} = \mathcal{B}_1 \cup \mathcal{B}_2$ . If  $v$  is a vertex of  $A$ , then the number of paths having  $v$  as end-vertex is exactly  $d_{F_1}(v)/2 + d_{M_2}(v)$ . Therefore, we have  $\mathcal{B}(v) = d_{F_1}(v)/2 + d_{M_2}(v) = d_{G_1}(v)/3 + d_{M_2}(v)$ . Analogously, we have  $\mathcal{B}(v) = d_{G_2}(v)/3 + d_{M_1}(v)$  for every vertex  $v$  in  $B$ . Thus,  $\mathcal{B}$  is an  $\mathbb{F}$ -balanced 3-path tracking decomposition of  $G$ . □

In the next lemma, we show how pre-completeness is related to completeness of odd tracking decompositions.

**Lemma 5.3.** *Let  $k$  be a positive integer and let  $G$  be a bipartite graph that admits a  $(1, 2k + 1)$ -bifactorization  $\mathbb{F}$ . If  $G$  admits an  $\mathbb{F}$ -balanced  $(2k + 1)$ -pre-complete  $(2k + 1)$ -tracking decomposition, then  $G$  admits an  $\mathbb{F}$ -balanced  $(k + 1)$ -complete  $(2k + 1)$ -tracking decomposition.*

*Proof.* Let  $k$  be a positive integer and let  $G = (A, B; E)$  be a bipartite graph that admits a  $(1, 2k + 1)$ -bifactorization  $\mathbb{F}$ . Let  $\mathcal{F}$  be the set of all  $\mathbb{F}$ -balanced  $(2k + 1)$ -pre-complete  $(2k + 1)$ -tracking decompositions of  $G$ . Now let  $\mathcal{B}$  be a tracking decomposition in  $\mathcal{F}$  such that  $\sum_{v \in V(G)} \text{hang}(v, \mathcal{B})$  is maximum over all tracking decompositions in  $\mathcal{F}$ . We claim that  $\mathcal{B}$  is a  $(k + 1)$ -complete tracking decomposition, i.e,  $\text{hang}(v, \mathcal{B}) > k + 1$  for each vertex  $v$  of  $G$ .

Suppose, for a contradiction, that  $\mathcal{B}$  is not  $(k + 1)$ -complete. Then there is a vertex  $v$  of  $G$  such that  $\text{hang}(v, \mathcal{B}) \leq k + 1$ . Suppose, without loss of generality, that  $v$  is a vertex of  $A$ . Since  $\mathcal{B}$  is  $(2k + 1)$ -pre-complete,  $\text{preHang}(v, \mathcal{B}) \geq 2k + 2$ . Thus, there are at least  $k + 1$  pre-hanging edges at  $v$  that are not hanging edges at  $v$ , say  $x_1 v, \dots, x_{k+1} v$ .

Let  $T_1 = y_0 y_1 \cdots y_{2k+1}$  be the element of  $\mathcal{B}$  that contains  $x_1 v$ , where, without loss of generality,  $y_0 = x_1$  and  $y_1 = v$ . The vertices of  $T_1$  in  $B$  are  $y_0, y_2, \dots, y_{2k}$ . First, we will show that  $x_i \notin V(T_1)$  for some  $2 \leq i \leq k+1$ . Since  $y_2$  is in  $B$  and  $y_0$  is the end-vertex of  $T_1$  in  $B$ , the vertex  $y_2$  is not an end-vertex of  $T_1$ . Since  $y_1 y_2 \in E(T_1)$ , the edge  $y_1 y_2$  is not a pre-hanging edge at  $v$ . Then,  $y_2 \neq x_i$  for every  $2 \leq i \leq k+1$ . Therefore,  $|\{y_4, y_6, \dots, y_{2k}\}| = k-1$  and  $|\{x_2, \dots, x_{k+1}\}| = k$ , from where we conclude that for at least one  $i$ , we have  $x_i \notin V(T_1)$ .

Now let  $T_i = z_0 z_1 \cdots z_{2k+1}$  be the element of  $\mathcal{B}$  that contains  $x_i v$ . Suppose, without loss of generality, that  $z_1 = v$  and  $z_0 = x_i$ . Let  $T'_1 = v y_1 \cdots y_{2k+1}$  and  $T'_i = y_0 z_1 \cdots z_{2k+1}$ , and let  $\mathcal{B}' = \mathcal{B} - T_1 - T_i + T'_1 + T'_i$ . We note that  $\bar{T}'_1 = \bar{T}_1 - x_1 v + x_i v$  and  $\bar{T}'_i = \bar{T}_i - x_i v + x_1 v$ . Since  $x_i \notin V(T_1)$ , we have  $d_{T'_1}(x_i) = 1$ , which implies that  $x_i v$  is a hanging edge of  $\mathcal{B}'$  at  $v$ . Therefore,  $\sum_{v \in V(G)} \text{hang}(v, \mathcal{B}') > \sum_{v \in V(G)} \text{hang}(v, \mathcal{B})$ .

We claim that  $\mathcal{B}'$  is  $\mathbb{F}$ -balanced. Indeed, since the set of pre-hanging edges at  $v$  is the same in  $\mathcal{B}$  and  $\mathcal{B}'$ , we have  $\mathcal{B}(v) = \mathcal{B}'(v)$  for every  $v$  in  $V(G)$ . Therefore,  $\mathcal{B}'$  is an  $\mathbb{F}$ -balanced  $(2k+1)$ -pre-complete tracking decomposition such that  $\sum_{v \in V(G)} \text{hang}(v, \mathcal{B}') > \sum_{v \in V(G)} \text{hang}(v, \mathcal{B})$ , a contradiction to the choice of  $\mathcal{B}$ .  $\square$

Now we are ready to use the Disentangling Lemma (Lemma 3.11) to obtain a path decomposition from the previous tracking decomposition.

**Lemma 5.4.** *Let  $k$  be a positive integer and let  $G$  be a bipartite graph that admits a  $(1, 2k+1)$ -bifactorization  $\mathbb{F}$ . If  $G$  admits an  $\mathbb{F}$ -balanced  $(k+1)$ -complete  $(2k+1)$ -tracking decomposition, then  $G$  admits an  $\mathbb{F}$ -balanced  $(k+1)$ -complete  $(2k+1)$ -path tracking decomposition.*

*Proof.* Let  $k$  be a positive integer and let  $G = (A, B; E)$  be a bipartite graph that admits a  $(1, 2k+1)$ -bifactorization  $\mathbb{F}$ . Suppose  $G$  admits an  $\mathbb{F}$ -balanced  $(k+1)$ -complete  $(2k+1)$ -tracking decomposition  $\mathcal{B}$ . By Lemma 3.11 (applied with  $\ell = 2k+1$ ), the graph  $G$  admits a  $(k+1)$ -complete  $(2k+1)$ -path tracking decomposition  $\mathcal{B}'$  such that  $\mathcal{B}'(v) = \mathcal{B}(v)$  for every vertex  $v$  of  $G$ . Therefore,  $\mathcal{B}'$  is an  $\mathbb{F}$ -balanced  $(k+1)$ -complete  $(2k+1)$ -path tracking decomposition.  $\square$

Now we have the tools needed for the proof of the next result, which is the main result of this section.

**Theorem 5.5.** *Let  $k$  be a positive integer. If  $G$  is a bipartite graph that admits a strong  $(1, 2k+1)$ -bifactorization  $\mathbb{F}$ , then  $G$  admits an  $\mathbb{F}$ -balanced  $(2k+1)$ -path tracking decomposition.*

*Proof.* The proof is by induction on  $k$ . By Lemma 5.2, the statement is true for  $k = 1$ . Thus, suppose  $k > 1$ , and let  $G = (A_1, A_2; E)$  be a bipartite graph that admits a strong  $(1, 2k+1)$ -bifactorization  $\mathbb{F} = (\mathcal{F}_1, \mathcal{F}_2)$ . We claim that  $G$  admits an  $\mathbb{F}$ -balanced  $(2k+1)$ -pre-complete  $(2k+1)$ -tracking decomposition. Let  $\mathcal{F}_1 = \{M_1, F_{1,1}, \dots, F_{1,k}\}$  and  $\mathcal{F}_2 = \{M_2, F_{2,2}, \dots, F_{2,k}\}$ , and let  $G_i = G[\bigcup_{F \in \mathcal{F}_i} F]$  for  $i = 1, 2$ . Hereafter, fix  $i \in \{1, 2\}$ .



Define  $d^*(v) = d_{G_i}(v)/(2k+1)$  for every vertex  $v$  in  $A_i$ . Note that  $d_{F_{i,j}}(v) = 2d^*(v) = 2d_{M_i}(v)$  for every vertex  $v$  in  $A_i$  and  $1 \leq j \leq k$ . Let  $O_{F_{i,k}}$  be a Eulerian orientation of  $F_{i,k}$ . Let  $F_{i,k} = F_{i,k}^{\text{forw}} \cup F_{i,k}^{\text{back}}$ , where  $F_{i,k}^{\text{forw}}$  is the set of edges of  $F_{i,k}$  leaving  $A_i$  in  $O_{F_{i,k}}$ , and  $F_{i,k}^{\text{back}}$  is the set of edges of  $F_{i,k}$  entering  $A_i$  in  $O_{F_{i,k}}$ . Note that, since  $O_{F_{i,k}}$  is an Eulerian orientation,  $d_{F_{i,k}^{\text{forw}}}(v) = d_{F_{i,k}^{\text{back}}}(v) = d_{F_{i,k}}(v)/2$  for every vertex  $v \in A_i$ .

Let  $G' = G - M_1 - M_2 - F_{1,k}^{\text{forw}} - F_{2,k}^{\text{forw}}$ , and define  $\mathcal{F}'_i = \{F_{i,k}^{\text{back}}, F_{i,1}, \dots, F_{i,k-1}\}$ . Let  $G'_i = G[\bigcup_{F \in \mathcal{F}'_i} F]$ . Note that  $G'_i = G_i - M_i - F_{i,k}^{\text{forw}}$ . Then, for every vertex  $v \in A_i$ , we have

$$d_{G'_i}(v) = d_{G_i}(v) - 2d^*(v) = (2k+1)d^*(v) - 2d^*(v) = (2k-1)d^*(v). \quad (4)$$

**Claim 5.6.**  $\mathbb{F}' = (\mathcal{F}'_1, \mathcal{F}'_2)$  is a strong  $(1, 2k-1)$ -bifactorization of  $G'$ .

*Proof.* To prove this claim, we shall prove the following.

- (i)  $F_{i,k}^{\text{back}}$  is an  $(A_i, 1, 2k-1)$ -factor of  $G'_i$ ;
- (ii)  $F_{i,j}$  is an Eulerian  $(A_i, 2, 2k-1)$ -factor of  $G'_i$  for  $1 \leq j \leq k-1$ ;
- (iii)  $d_{G'_i}(v) \geq (2k-1)(2k)$  for every vertex  $v \in A_i$ .

To prove items (i) and (ii), first note that, for every  $v \in A_i$ , we have  $d_{F_{i,k}^{\text{back}}}(v) = d^*(v)$  and  $d_{F_{i,j}}(v) = 2d^*(v)$  for every  $1 \leq j \leq k-1$ . By (4), we conclude that  $F_{i,k}^{\text{back}}$  is a  $(A_i, 1, 2k-1)$ -factor of  $G'_i$  and  $F_{i,j}$  is an  $(A_i, 2, 2k-1)$ -factor of  $G'_i$ . Since  $\mathbb{F}$  is a  $(1, 2k+1)$ -bifactorization,  $F_{1,j}$  and  $F_{2,j}$  are Eulerian graphs for  $1 \leq j \leq k-1$ .

It remains to prove item (iii). Since  $d_{G'_i}(v) = (2k-1)d^*(v)$  and  $d^*(v) = d_{G_i}(v)/(2k+1)$  for every vertex  $v \in A_i$ , we have  $d_{G'_i}(v) = \frac{2k-1}{2k+1}d_{G_i}(v)$  for every  $v \in A_i$ . Since  $\mathbb{F}$  is a strong  $(1, 2k+1)$ -bifactorization,  $d_{G_i}(v) \geq (2k+1)(2k+2)$  for every  $v \in A_i$ . Therefore,  $d_{G'_i}(v) \geq (2k-1)(2k+2) > (2k-1)2k$  for every  $v \in A_i$ .  $\square$

Since  $\mathbb{F}'$  is a strong  $(1, 2k-1)$ -bifactorization of  $G'$ , by the induction hypothesis,  $G'$  admits an  $\mathbb{F}'$ -balanced  $(2k-1)$ -path tracking decomposition  $\mathcal{B}'$ .

Since  $\mathcal{B}'$  is a  $\mathbb{F}'$ -balanced  $(2k-1)$ -path tracking decomposition, we have

- $\mathcal{B}'(v) = d_{G'_1}(v)/(2k-1) + d_{F_{2,k}^{\text{back}}}(v)$  for every  $v \in A_1$ ;
- $\mathcal{B}'(v) = d_{G'_2}(v)/(2k-1) + d_{F_{1,k}^{\text{back}}}(v)$  for every  $v \in A_2$ .

Now we want to extend each tracking in  $\mathcal{B}'$  to obtain a  $(2k+1)$ -tracking decomposition of  $G$ . For that, we add edges from  $E(G) - E(G')$  at the end-vertices of the trackings in  $\mathcal{B}'$ . For each  $v \in A_1$  ( $v \in A_2$ ), let  $S_v$  be the set of edges of  $M_1 \cup F_{2,k}^{\text{forw}}$  ( $M_2 \cup F_{1,k}^{\text{forw}}$ ) that are incident to  $v$ . Note that for each edge  $e$  in  $E(G) - E(G')$  there is exactly one  $v \in V(G)$  such that  $e \in S_v$ . Then,  $\bigcup_{v \in V(G)} S_v = E(G) - E(G')$ . Therefore, if we prove that  $\mathcal{B}'(v) = |S_v|$ , then we can extend every tracking  $B$  in  $\mathcal{B}'$  by adding one edge at each end-vertex of  $B$ .

**Claim 5.7.**  $\mathcal{B}'(v) = |S_v|$ , for every  $v \in V(G)$ .

*Proof.* First, note that, since  $F_{i,k}$  is Eulerian, we have  $d_{F_{2,k}^{\text{back}}}(v) = d_{F_{2,k}^{\text{forw}}}(v)$  for every  $v \in A_1$ , and  $d_{F_{1,k}^{\text{back}}}(v) = d_{F_{1,k}^{\text{forw}}}(v)$  for every  $v \in A_2$ .

For every  $v \in A_i$  and every  $1 \leq j \leq k-1$ , we have  $d_{G'_i}(v)/(2k-1) = d_{F_{i,j}}(v)/2 = d^*(v)$ . Now, recall that for each  $v \in A_i$ , we have  $d_{M_i}(v) = d^*(v)$ . Therefore, for every  $v \in A_1$ , we have

$$\mathcal{B}'(v) = d_{G'_1}(v)/(2k-1) + d_{F_{2,k}^{\text{back}}}(v) = d^*(v) + d_{F_{2,k}^{\text{forw}}}(v) = d_{M_1}(v) + d_{F_{2,k}^{\text{forw}}}(v) = |S_v|.$$

Similarly, we have  $\mathcal{B}'(v) = |S_v|$  for every  $v \in A_2$ .  $\square$

As we have seen, we can extend every tracking  $B$  of  $\mathcal{B}'$  by adding one edge at each end-vertex of  $B$ . Let  $\mathcal{B}$  be the tracking decomposition obtained by this extension. Since  $\mathcal{B}'$  is a  $(2k-1)$ -path tracking decomposition and we added edges only at the end-vertices of these trackings,  $\mathcal{B}$  is composed of trackings of vanilla  $(2k+1)$ -trails. Furthermore, since we added all edges in  $E(G) - E(G')$ ,  $\mathcal{B}$  is a decomposition of  $G$ .

**Claim 5.8.**  $\mathcal{B}$  is  $\mathbb{F}$ -balanced.

*Proof.* Fix  $x_0 \in A_1$ . First we will prove that  $\mathcal{B}(x_0) \leq d_{F_{1,k}^{\text{forw}}}(x_0) + d_{M_2}(x_0)$ . If there is no tracking  $T = x_0x_1 \cdots x_{2k+1}$  in  $\mathcal{B}$ , where  $x_0x_1$  is an edge of  $E(G) - E(G')$ , then  $\mathcal{B}(x_0) = 0$ . For every such tracking  $T$ , by the construction of  $\mathcal{B}$ , we know that  $x_0x_1$  is an element of  $S_{x_1}$ . Since  $x_1$  is a vertex of  $A_2$ , we have  $S_{x_1} \subset M_2 \cup F_{1,k}^{\text{forw}}$ . Therefore,

$$\mathcal{B}(x_0) \leq d_{F_{1,k}^{\text{forw}}}(x_0) + d_{M_2}(x_0).$$

Now we will prove that  $\mathcal{B}(x_0) \geq d_{F_{1,k}^{\text{forw}}}(x_0) + d_{M_2}(x_0)$ . Note that, if  $x_0x_1$  is an edge of  $M_2 \cup F_{1,k}^{\text{forw}}$  that is incident to  $x_0$  in  $A_1$  (these are the only edges of  $G$  that can contribute to  $\mathcal{B}(x_0)$ ), then, by the construction of  $\mathcal{B}$ , there is a tracking  $Q' = x_1 \cdots x_{2k}$  in  $\mathcal{B}'$  of a path such that the tracking  $Q = x_0x_1 \cdots x_{2k+1}$  (of a vanilla trail) belongs to  $\mathcal{B}$ . Therefore  $\mathcal{B}(x_0) \geq d_{F_{1,k}^{\text{forw}}}(x_0) + d_{M_2}(x_0)$ , from where we conclude that  $\mathcal{B}(v) = d_{F_{1,k}^{\text{forw}}}(v) + d_{M_2}(v)$  for every  $v$  in  $A_1$ . Thus, for every vertex  $v$  in  $A_1$  we have

$$\mathcal{B}(v) = d_{F_{1,k}^{\text{forw}}}(v) + d_{M_2}(v) = d^*(v) + d_{M_2}(v) = d_{G_1}(v)/(2k+1) + d_{M_2}(v).$$

Analogously, we have  $\mathcal{B}(v) = d_{G_2}(v)/(2k+1) + d_{M_1}(v)$  for each vertex  $v$  in  $A_2$ . Thus,  $\mathcal{B}$  is an  $\mathbb{F}$ -balanced  $(2k+1)$ -tracking decomposition.  $\square$

**Claim 5.9.**  $\mathcal{B}$  is  $(2k+1)$ -pre-complete.

*Proof.* Let  $v \in A_i$ . We shall prove that  $\text{preHang}(v, \mathcal{B}) > 2k+1$ . Note that, by the construction of  $\mathcal{B}$ , the set of pre-hanging edges at  $v$  in  $\mathcal{B}$  is exactly  $S_v$ . Then,  $\text{preHang}(v, \mathcal{B}) = |S_v| = \mathcal{B}'(v)$ . Since  $\mathcal{B}'$  is balanced,  $\mathcal{B}'(v) \geq d_{G'_i}(v)/(2k-1)$ . Therefore,

$$\text{preHang}(v, \mathcal{B}) = \mathcal{B}'(v) \geq d_{G'_i}(v)/(2k-1) = d^*(v) = d_{G_i}(v)/(2k+1).$$

Since  $\mathbb{F}$  is a strong  $(1, 2k+1)$ -bifactorization of  $G$ , we have  $d_{G_i}(v) \geq (2k+1)(2k+2)$ , from where we conclude that  $\text{preHang}(v, \mathcal{B}) \geq 2k+2$ . Therefore,  $\mathcal{B}$  is a  $(2k+1)$ -pre-complete tracking decomposition.  $\square$

Now we are able to conclude the proof. By Lemma 5.3,  $G$  admits an  $\mathbb{F}$ -balanced  $(k+1)$ -complete  $(2k+1)$ -tracking decomposition. By Lemma 5.4,  $G$  admits an  $\mathbb{F}$ -balanced  $(k+1)$ -complete  $(2k+1)$ -path tracking decomposition. Therefore,  $G$  admits an  $\mathbb{F}$ -balanced  $(2k+1)$ -path tracking decomposition.  $\square$

## 6. DECOMPOSITION INTO PATHS OF EVEN LENGTH

The technique used in this section is analogous to the one used in Section 5. The results are similar, but to deal with paths of even length some adjustments were necessary.

**Definition 6.1** (Balanced tracking decomposition – even case). *Let  $k$  be a positive integer. Let  $G = (A, B; E)$  be a bipartite graph that admits a  $(2, 2(2k+2))$ -bifactorization  $\mathbb{F} = (\mathcal{F}_1, \mathcal{F}_2)$ , and let  $G_i = G[\bigcup_{F \in \mathcal{F}_i} F]$  for  $i = 1, 2$ . Let  $M_1, N_1$  be the  $(A, 1, 2(2k+2))$ -factors of  $\mathbb{F}$  and  $M_2, N_2$  be the  $(B, 1, 2(2k+2))$ -factors of  $\mathbb{F}$ . We say that a  $(2k+2)$ -tracking decomposition  $\mathcal{B}$  of  $G$  into is  $\mathbb{F}$ -balanced if the following holds.*

- $\mathcal{B}(v) = d_{G_1}(v)/(2k+2) + d_{M_2}(v) + d_{N_2}(v)$  for every  $v \in A$ ;
- $\mathcal{B}(v) = d_{G_2}(v)/(2k+2) + d_{M_1}(v) + d_{N_1}(v)$  for every  $v \in B$ .

Our aim is to prove Theorem 6.5, which guarantees that one may obtain an  $\mathbb{F}$ -balanced  $(2k+2)$ -complete  $(2k+2)$ -path tracking decomposition from a strong  $(2, 2(2k+2))$ -bifactorization  $\mathbb{F}$ . First, we show that from a  $(2, 4)$ -bifactorization we may obtain a balanced 2-path tracking decomposition.

**Lemma 6.2.** *If  $G$  is a bipartite graph that admits a  $(2, 4)$ -bifactorization  $\mathbb{F}$ , then  $G$  admits an  $\mathbb{F}$ -balanced 2-path decomposition.*

*Proof.* Let  $G = (A, B; E)$  be a bipartite graph that admits a  $(2, 4)$ -bifactorization  $\mathbb{F} = (\mathcal{F}_1, \mathcal{F}_2)$  and put  $\mathcal{F}_i = \{M_i, N_i, F_i\}$  for  $i = 1, 2$ . Let  $O_{F_i}$  be an Eulerian orientation of  $G[F_i]$ , for  $i = 1, 2$ . Let  $\mathcal{C}_1$  be the set of components of  $G[F_1]$ . Let  $T$  be an element of  $\mathcal{C}_1$  and  $B_T = a_0 b_0 a_1 b_1 \cdots a_s b_s$  be a tracking of  $T$ , where  $a_i \in A$ , and  $b_i \in B$ , for  $1 \leq i \leq s$ . We have that  $\mathcal{B}'_T = \{a_i b_i a_{i+1} : 0 \leq i \leq s\}$ , taking  $a_{s+1} = a_0$ , is a 2-tracking decomposition of  $T$  in which every tracking has its end-vertices in  $A$ . Therefore,  $\mathcal{B}'_1 = \cup_{T \in \mathcal{C}_1} \mathcal{B}'_T$  is a 2-tracking decomposition of  $G[F_1]$  in which every tracking has its end-vertices in  $A$ . Analogously  $G[F_2]$  admits a 2-tracking decomposition  $\mathcal{B}'_2$  in which every tracking has its end-vertices in  $B$ .

Let  $v$  be a vertex in  $A$ . Since  $M_1$  and  $N_1$  are  $(1, 1, 4)$ -factors of  $G$  we have  $d_{M_1}(v) = d_{N_1}(v)$ . Thus, the number of edges in  $M_1 \cup N_1$  incident to  $v$  is even, and we can decompose the edges in  $M_1 \cup N_1$  incident to  $v$  into 2-paths such that each path has its end-vertices in  $B$ . Taking any tracking of each of these paths, we obtain a 2-tracking decomposition  $\mathcal{B}''_1$  of the edges in  $M_1 \cup N_1$  such that each path has its end-vertices in  $B$ . Analogously, there is a 2-tracking decomposition  $\mathcal{B}''_2$  of the edges in  $M_2 \cup N_2$  such that each path has its end-vertices in  $A$ .

Let  $\mathcal{B} = \mathcal{B}'_1 \cup \mathcal{B}'_2 \cup \mathcal{B}''_1 \cup \mathcal{B}''_2$ . Note that only the paths in  $\mathcal{B}'_1$  and in  $\mathcal{B}''_2$  have end-vertices in  $A$ , and analogously only the paths in  $\mathcal{B}'_2$  and in  $\mathcal{B}''_1$  have end-vertices in  $B$ . Therefore, if  $v$  is a vertex of  $A$ , then  $\mathcal{B}(v) = \mathcal{B}'_1(v) + \mathcal{B}''_2(v) = d_G(v)/2 + d_{M_2}(v) + d_{N_2}(v)$ , and if  $v$  is a vertex of  $B$ , then  $\mathcal{B}(v) = \mathcal{B}'_2(v) + \mathcal{B}''_1(v) = d_G(v)/2 + d_{M_1}(v) + d_{N_1}(v)$ . Thus,  $\mathcal{B}$  is an  $\mathbb{F}$ -balanced 2-path tracking decomposition of  $G$ .  $\square$

In the next lemma, we show how pre-completeness is related to completeness of even tracking decompositions.

**Lemma 6.3.** *Let  $k$  be a positive integer. Let  $G$  be a bipartite graph that admits a  $(2, 2(2k+2))$ -bifactorization  $\mathbb{F}$ . If  $G$  admits an  $\mathbb{F}$ -balanced  $(2k+3)$ -pre-complete  $(2k+2)$ -tracking decomposition, then  $G$  admits an  $\mathbb{F}$ -balanced  $(k+2)$ -complete  $(2k+2)$ -tracking decomposition.*

*Proof.* Let  $k$  be a positive integer. Let  $G = (A, B; E)$  be a bipartite graph that admits a  $(2, 2(2k+2))$ -bifactorization  $\mathbb{F}$ . Let  $\mathcal{F}$  be the set of all  $\mathbb{F}$ -balanced  $(2k+3)$ -pre-complete  $(2k+2)$ -tracking decompositions of  $G$ . Now let  $\mathcal{B}$  be a tracking decomposition in  $\mathcal{F}$  such that  $\sum_{v \in V(G)} \text{hang}(v, \mathcal{B})$  is maximum over all tracking decompositions in  $\mathcal{F}$ . We claim that  $\mathcal{B}$  is a  $(k+2)$ -complete tracking decomposition, i.e,  $\text{hang}(v, \mathcal{B}) > k+2$  for each vertex  $v$  of  $G$ .

Suppose, for a contradiction, that  $\mathcal{B}$  is not  $(k+2)$ -complete. Then there is a vertex  $v$  of  $G$  such that  $\text{hang}(v, \mathcal{B}) \leq k+2$ . Suppose, without loss of generality, that  $v$  is a vertex of  $A$ . Since  $\mathcal{B}$  is  $(2k+3)$ -pre-complete,  $\text{preHang}(v, \mathcal{B}) \geq 2k+4$ . Thus, there are at least  $k+2$  pre-hanging edges at  $v$  that are not hanging edges at  $v$ , say  $x_1v, \dots, x_{k+2}v$ . Let  $T_1 = y_0y_1 \cdots y_{2k+2}$  be the element of  $\mathcal{B}$  that contains  $x_1v$ , where, without loss of generality,  $y_0 = x_1$  and  $y_1 = v$ . The vertices of  $T_1$  in  $B$  are  $y_0, y_2, \dots, y_{2k+2}$ . First, we will show that for some  $x_i \notin V(T_1)$  for some  $2 \leq i \leq k+2$ . Since  $y_2$  is in  $B$  and  $y_0$  is the end-vertex of  $T_1$  in  $B$ , the vertex  $y_2$  is not an end-vertex of  $T_1$ . Since  $y_1y_2 \in E(T_1)$ , the edge  $y_1y_2$  is not a pre-hanging edge at  $v$ . Then,  $y_2 \neq x_i$  for every  $2 \leq i \leq k+2$ . Therefore,  $|\{y_4, y_6, \dots, y_{2k+2}\}| = k$  and  $|\{2, \dots, k+2\}| = k+1$ , from where we conclude that for at least one  $i$ , we have  $x_i \notin V(T_1)$ .

Now let  $T_i = z_0z_1 \cdots z_{2k+2}$  be the element of  $\mathcal{B}$  that contains  $x_iv$ . Suppose, without loss of generality, that  $z_1 = v$  and  $z_0 = x_i$ . Let  $T'_1 = vy_1 \cdots y_{2k+1}$  and  $T'_i = y_0z_1 \cdots z_{2k+2}$  and let  $\mathcal{B}' = \mathcal{B} - T_1 - T_i + T'_1 + T'_i$ . We note that  $\bar{T}'_1 = \bar{T}_1 - x_1v + x_iv$  and  $\bar{T}'_i = \bar{T}_i - x_iv + x_1v$ . Since  $x_i \notin V(T_1)$ , we have  $d_{T'_1}(x_i) = 1$ , which implies that  $x_iv$  is a hanging edge of  $\mathcal{B}'$  at  $v$ . Therefore,  $\sum_{v \in V(G)} \text{hang}(v, \mathcal{B}') > \sum_{v \in V(G)} \text{hang}(v, \mathcal{B})$ .

We claim that  $\mathcal{B}'$  is  $\mathbb{F}$ -balanced. Indeed, since the set of pre-hanging edges at  $v$  is the same in  $\mathcal{B}$  and  $\mathcal{B}'$ , we have  $\mathcal{B}(v) = \mathcal{B}'(v)$  for every  $v$  in  $V(G)$ . Therefore,  $\mathcal{B}'$  is an  $\mathbb{F}$ -balanced  $(2k+3)$ -pre-complete decomposition such that  $\sum_{v \in V(G)} \text{hang}(v, \mathcal{B}') > \sum_{v \in V(G)} \text{hang}(v, \mathcal{B})$ , a contradiction to the choice of  $\mathcal{B}$ .  $\square$

As we did for the odd paths, we use the Disentangling Lemma to obtain a path decomposition.

**Lemma 6.4.** *Let  $k$  be a positive integer and let  $G$  be a bipartite graph that admits a  $(2, 2(2k+2))$ -bifactorization  $\mathbb{F}$ . If  $G$  admits an  $\mathbb{F}$ -balanced  $(k+2)$ -complete vanilla  $(2k+2)$ -tracking decomposition, then  $G$  admits an  $\mathbb{F}$ -balanced  $(k+2)$ -complete  $(2k+2)$ -path tracking decomposition.*

*Proof.* Let  $k$  be a positive integer and let  $G = (A, B; E)$  be a bipartite graph that admits a  $(2, 2(2k+2))$ -bifactorization  $\mathbb{F}$ . Suppose  $G$  admits an  $\mathbb{F}$ -balanced  $(k+2)$ -complete  $(2k+2)$ -tracking decomposition  $\mathcal{B}$ . By Lemma 3.11 (applied with  $\ell = 2k+2$ ), the graph  $G$  admits a  $(k+2)$ -complete tracking decomposition  $\mathcal{B}'$  such that  $\mathcal{B}'(v) = \mathcal{B}(v)$  for every vertex  $v$  of  $G$ . Therefore,  $\mathcal{B}'$  is an  $\mathbb{F}$ -balanced  $(k+2)$ -complete  $(2k+2)$ -tracking decomposition.  $\square$

Now we are ready to prove the main result of this section.

**Theorem 6.5.** *Let  $k$  be a positive integer. If  $G$  is a bipartite graph that admits a strong  $(2, 2(2k+2))$ -bifactorization  $\mathbb{F}$ , then  $G$  admits an  $\mathbb{F}$ -balanced  $(2k+2)$ -path tracking decomposition.*

*Proof.* The proof is by induction on  $k$ . By Lemma 6.2, the statement is true for  $k = 0$ . Thus, suppose  $k > 0$ , and let  $G = (A_1, A_2; E)$  be a bipartite graph that admits a strong  $(2, 2(2k+2))$ -bifactorization  $\mathbb{F} = (\mathcal{F}_1, \mathcal{F}_2)$ . We claim that  $G$  admits an  $\mathbb{F}$ -balanced  $(2k+3)$ -pre-complete vanilla  $(2k+2)$ -trail decomposition. Let  $\mathcal{F}_1 = \{M_1, N_1, F_{1,1}, \dots, F_{1,2k+1}\}$  and  $\mathcal{F}_2 = \{M_2, N_2, F_{2,1}, \dots, F_{2,2k+1}\}$ , and let  $G_i = G[\bigcup_{F \in \mathcal{F}_i} F]$  for  $i = 1, 2$ . Hereafter, fix  $i \in \{1, 2\}$ .

Define  $d^*(v) = d_{G_i}(v)/(2(2k+2))$  for every vertex  $v \in A_i$ . Note that  $d_{F_{i,j}}(v) = 2d^*(v) = 2d_{M_i}(v) = 2d_{N_i}(v)$  for every vertex  $v$  in  $A_i$  and  $1 \leq j \leq 2k+1$ . For  $j \in \{2k, 2k+1\}$ , let  $O_{F_{i,j}}$  be an Eulerian orientation of  $G[F_{i,j}]$ . Let  $F_{i,j} = F_{i,j}^{\text{forw}} \cup F_{i,j}^{\text{back}}$ , where  $F_{i,j}^{\text{forw}}$  is the set of edges of  $F_{i,j}$  leaving  $A_i$  in  $O_{F_{i,j}}$ , and  $F_{i,j}^{\text{back}}$  is the set of edges of  $F_{i,j}$  entering  $A_i$  in  $O_{F_{i,j}}$ .

Let  $G' = G - M_1 - N_1 - M_2 - N_2 - F_{1,2k}^{\text{forw}} - F_{1,2k+1}^{\text{forw}} - F_{2,2k}^{\text{forw}} - F_{2,2k+1}^{\text{forw}}$ , and let  $\mathcal{F}'_i = \{F_{i,2k}^{\text{back}}, F_{i,2k+1}^{\text{back}}, F_{i,1}, \dots, F_{i,2k-1}\}$ . Let  $G'_i = G[\bigcup_{F \in \mathcal{F}'_i} F]$ . Note that  $G'_i = G_i - M_i - N_i - F_{i,2k}^{\text{forw}} - F_{i,2k+1}^{\text{forw}}$ . Then, for every  $v \in A_i$ , we have

$$d_{G'_i}(v) = d_{G_i}(v) - 4d^*(v) = 2(2k+2)d^*(v) - 4d^*(v) = 2(2k)d^*(v). \quad (5)$$

**Claim 6.6.**  $\mathbb{F}' = (\mathcal{F}'_1, \mathcal{F}'_2)$  is a strong  $(2, 2(2k))$ -bifactorization of  $G'$ .

*Proof.* To prove this claim, we shall prove the following.

- (i)  $F_{i,2k}^{\text{back}}$  and  $F_{i,2k+1}^{\text{back}}$  are  $(A_i, 1, 2(2k))$ -factors of  $G'_i$ ;
- (ii)  $F_{i,j}$  is an Eulerian  $(A_i, 2, 2(2k))$ -factor of  $G'_i$  for  $j = 1, \dots, 2k-1$ ;

(iii)  $d_{G'_i}(v) \geq (2k)(2k+2)$  for every vertex  $v \in A_i$ .

To prove items (i) and (ii), first note that, for every  $v \in A_i$ , we have  $d_{F_{i,2k}^{\text{back}}}(v) = d_{F_{i,2k+1}^{\text{back}}}(v) = d^*(v)$  and  $d_{F_{i,j}}(v) = 2d^*(v)$  for every  $1 \leq j \leq 2k-1$ . By (5), we conclude that  $F_{i,2k}^{\text{back}}$  and  $F_{i,2k+1}^{\text{back}}$  are  $(A_i, 1, 2(2k))$ -factors of  $G'_i$ , and  $F_{i,j}$  is an  $(A_i, 2, 2(2k))$ -factor of  $G'_i$ . Since  $\mathbb{F}$  is a  $(2, 2(2k+2))$ -bifactorization,  $F_{1,j}$  and  $F_{2,j}$  are Eulerian graphs for  $1 \leq j \leq 2k-1$ .

It remains to prove item (iii). Since  $d_{G'_i}(v) = 2(2k)d^*(v)$  and  $d^*(v) = d_{G_i}(v)/(2(2k+2))$  for every vertex  $v \in A_i$ , we have  $d_{G'_i}(v) = \frac{2(2k)}{2(2k+2)}d_{G_1}(v)$  for every  $v \in A_i$ . Since  $\mathbb{F}$  is a strong  $(2, 2(2k+2))$ -bifactorization, we have  $d_{G_i}(v) \geq (2k+2)(2k+4)$  for every  $v \in A_i$ . Thus  $d_{G'_i}(v) \geq (2k)(2k+4) > (2k)(2k+2)$  for every  $v \in A_i$ .  $\square$

Since  $\mathbb{F}'$  is a strong  $(2, 2(2k))$ -bifactorization of  $G'$ , by the induction hypothesis,  $G'$  admits an  $\mathbb{F}'$ -balanced  $2k$ -path tracking decomposition  $\mathcal{B}'$ .

Since  $\mathcal{B}'$  is an  $\mathbb{F}'$ -balanced  $2k$ -path tracking decomposition, we have

- $\mathcal{B}'(v) = d_{G'_1}(v)/2k + d_{F_{2,2k}^{\text{back}}}(v) + d_{F_{2,2k+1}^{\text{back}}}(v)$  for every  $v \in A_1$ ;
- $\mathcal{B}'(v) = d_{G'_2}(v)/2k + d_{F_{1,2k}^{\text{back}}}(v) + d_{F_{1,2k+1}^{\text{back}}}(v)$  for every  $v \in A_2$ .

Now we want to extend each  $2k$ -path of  $\mathcal{B}'$  to obtain a  $(2k+2)$ -tracking decomposition of  $G$ . For that, we add edges from  $E(G) - E(G')$  at the end-vertices of the trackings in  $\mathcal{B}'$ . For each vertex  $v \in A_1$  ( $v \in A_2$ ), let  $S_v$  be the set of edges of  $M_1 \cup N_1 \cup F_{2,2k}^{\text{forw}} \cup F_{2,2k+1}^{\text{forw}}$  ( $M_2 \cup N_2 \cup F_{1,2k}^{\text{forw}} \cup F_{1,2k+1}^{\text{forw}}$ ) incident to  $v$ . Note that for each edge  $e \in E(G) - E(G')$  there is exactly one  $v \in V(G)$  such that  $e \in S_v$ . Then,  $\bigcup_{v \in V(G)} S_v = E(G) - E(G')$ . Therefore, if we prove that  $\mathcal{B}'(v) = |S_v|$ , then we can extend every tracking  $B$  of  $\mathcal{B}'$  by adding one edge at each end-vertex of  $B$ .

**Claim 6.7.**  $\mathcal{B}'(v) = |S_v|$ , for every  $v \in V(G)$ .

*Proof.* First, note that, since  $F_{2,2k}$  and  $F_{2,2k+1}$  are Eulerian, we have  $d_{F_{2,2k}^{\text{back}}}(v) = d_{F_{2,2k}^{\text{forw}}}(v)$  and  $d_{F_{2,2k+1}^{\text{back}}}(v) = d_{F_{2,2k+1}^{\text{forw}}}(v)$  for every vertex  $v$  in  $A_1$ , and since  $F_{1,2k}$  and  $F_{1,2k+1}$  are Eulerian, we have  $d_{F_{1,2k}^{\text{back}}}(v) = d_{F_{1,2k}^{\text{forw}}}(v)$  and  $d_{F_{1,2k+1}^{\text{back}}}(v) = d_{F_{1,2k+1}^{\text{forw}}}(v)$  for every vertex  $v$  in  $A_2$ .

For every  $v \in A_i$  and every  $1 \leq j \leq 2k-1$ , we have  $d_{G'_i}(v)/(2(2k)) = d_{F_{i,j}}(v)/2 = d^*(v)$ . Now, recall that for each vertex  $v \in A_i$ , we have  $d_{M_i}(v) = d_{N_i}(v) = d^*(v)$ . Therefore, for every  $v \in A_1$ , we have

$$\begin{aligned} \mathcal{B}'(v) &= d_{G'_1}(v)/(2k) + d_{F_{2,2k}^{\text{back}}}(v) + d_{F_{2,2k+1}^{\text{back}}}(v) \\ &= 2d^*(v) + d_{F_{2,2k}^{\text{forw}}}(v) + d_{F_{2,2k+1}^{\text{forw}}}(v) \\ &= d_{M_1}(v) + d_{N_1}(v) + d_{F_{2,2k}^{\text{forw}}}(v) + d_{F_{2,2k+1}^{\text{forw}}}(v) \\ &= |S_v|. \end{aligned}$$

Similarly, we have  $\mathcal{B}'(v) = |S_v|$  for every  $v \in A_2$ .  $\square$

We have shown that every tracking  $B$  of  $\mathcal{B}'$  can be extended by adding one edge at each of its end-vertices. Let  $\mathcal{B}$  be the decomposition obtained with these extensions. Analogously to the odd case, we conclude that  $\mathcal{B}$  is a  $(2k+2)$ -tracking decomposition of  $G$ .

**Claim 6.8.**  $\mathcal{B}$  is  $\mathbb{F}$ -balanced.

*Proof.* Let  $x_0$  be a vertex in  $A$ . First we will prove that  $\mathcal{B}(x_0) \leq d_{F_{1,2k}^{\text{forw}}}(x_0) + d_{F_{1,2k+1}^{\text{forw}}}(x_0) + d_{M_2}(x_0) + d_{N_2}(x_0)$ . If there is no tracking  $T = x_0x_1 \cdots x_{2k+2}$  in  $\mathcal{B}$ , where  $x_0x_1$  is an edge of  $E(G) - E(G')$ , then  $\mathcal{B}(x_0) = 0$ . For every such tracking  $T$ , by the construction of  $\mathcal{B}$ , we know that  $x_0x_1$  is an element of  $S_{x_1}$ . Since  $x_1$  is a vertex of  $A_2$ , we have that  $S_{x_1} \subset M_2 \cup N_2 \cup F_{1,2k}^{\text{forw}} \cup F_{1,2k+1}^{\text{forw}}$ . Therefore,

$$\mathcal{B}(x_0) \leq d_{F_{1,2k}^{\text{forw}}}(x_0) + d_{F_{1,2k+1}^{\text{forw}}}(x_0) + d_{M_2}(x_0) + d_{N_2}(x_0).$$

Now we will prove that  $\mathcal{B}(x_0) \geq d_{F_{1,2k}^{\text{forw}}}(x_0) + d_{F_{1,2k+1}^{\text{forw}}}(x_0) + d_{M_2}(x_0) + d_{N_2}(x_0)$ . Note that if  $x_0x_1$  is an edge of  $M_2 \cup N_2 \cup F_{1,2k}^{\text{forw}} \cup F_{1,2k+1}^{\text{forw}}$  that is incident to  $x_0$  in  $A_1$  (these are the only edges of  $G$  that can contribute to  $\mathcal{B}(x_0)$ ), then, by the construction of  $\mathcal{B}$ , there is a tracking  $Q' = x_1 \cdots x_{2k+2}$  of a path such that the tracking  $Q = x_0x_1 \cdots x_{2k+2}x_0$  (of a vanilla trail) belongs to  $\mathcal{B}$ . Therefore,  $\mathcal{B}(x_0) = d_{F_{1,2k}^{\text{forw}}}(x_0) + d_{F_{1,2k+1}^{\text{forw}}}(x_0) + d_{M_2}(x_0) + d_{N_2}(x_0)$ . Thus, for every vertex  $v \in A_1$  we have

$$\begin{aligned} \mathcal{B}(v) &= |F_{1,2k}^{\text{forw}}(v)| + |F_{1,2k+1}^{\text{forw}}(v)| + |M_2(v)| + |N_2(v)| \\ &= 2d^*(v) + d_{M_2}(v) + d_{N_2}(v) \\ &= d_{G_1}(v)/(2k+2) + d_{M_2}(v) + d_{N_2}(v). \end{aligned}$$

Analogously, we have  $\mathcal{B}(v) = d_{G_2}(v)/(2k+2) + d_{M_1}(v) + d_{N_1}(v)$  for each vertex  $v \in A_2$ . Thus,  $\mathcal{B}$  is an  $\mathbb{F}$ -balanced vanilla  $(2k+2)$ -trail decomposition.  $\square$

**Claim 6.9.**  $\mathcal{B}$  is  $(2k+3)$ -pre-complete.

*Proof.* Let  $v \in A_i$ . We shall prove that  $\text{preHang}(v, \mathcal{B}) > 2k+3$ . Note that, by the construction of  $\mathcal{B}$ , the set of pre-hanging edges at  $v$  in  $\mathcal{B}$  is exactly  $S_v$ . Then,  $\text{preHang}(v, \mathcal{B}) = |S_v| = \mathcal{B}'(v)$ . Since  $\mathcal{B}'$  is balanced,  $\mathcal{B}'(v) \geq d_{G'_i}(v)/(2k)$ . Therefore,

$$\text{preHang}(v, \mathcal{B}) = \mathcal{B}'(v) \geq d_{G'_i}(v)/(2k) = 2d^*(v) = d_{G_i}(v)/(2k+2).$$

Since  $\mathbb{F}$  is a strong  $(2, 2(2k+2))$ -bifactorization of  $G$ , we have  $d_{G_i}(v) \geq (2k+2)(2k+4)$ , from where we conclude that  $\text{preHang}(v, \mathcal{B}) \geq 2k+4$ . Therefore,  $\mathcal{B}$  is a  $(2k+3)$ -pre-complete vanilla trail decomposition.  $\square$

Now, analogously to the odd case, using Lemmas 6.3 and 6.4, we conclude that  $G$  admits an  $\mathbb{F}$ -balanced  $(2k+2)$ -path tracking decomposition.  $\square$

## 7. DECOMPOSITION OF HIGHLY EDGE-CONNECTED GRAPHS INTO PATHS

In this section we put together the results of Section 4 and Theorem 5.5 (resp. Theorem 6.5) and prove Conjecture 1.2 for paths of odd (resp. even) length.

### 7.1. Paths of odd length.

**Theorem 7.1.** *Let  $k$  be a positive integer and let  $r = (2k + 1)(2k + 2)$ . If  $G$  is a  $2(6k + 2r + 1)$ -edge-connected bipartite graph such that  $|E(G)|$  is divisible by  $2k + 1$ , then  $G$  admits a decomposition into  $(2k + 1)$ -paths.*

*Proof.* Let  $k$  be a positive integer and let  $G$  be a bipartite graph such that  $|E(G)|$  is divisible by  $2k + 1$ . Let  $r = (2k + 1)(2k + 2)$ . Suppose  $G$  is a  $2(6k + 2r + 1)$ -edge-connected graph. By Lemma 4.8,  $G$  is strongly  $(1, 2k + 1)$ -bifactorable. Let  $\mathbb{F}$  be a strong  $(1, 2k + 1)$ -bifactorization of  $G$ . By Theorem 5.5,  $G$  admits an  $\mathbb{F}$ -balanced  $(2k + 1)$ -path decomposition.  $\square$

**7.2. Paths of even length.** The proof of this case is slightly different from the odd case. First, we prove that Conjecture 1.2 is equivalent to the following conjecture.

**Conjecture 7.2.** *For each tree  $T$ , there exists a positive integer  $k_T''$  such that, if  $G$  is a  $k_T''$ -edge-connected bipartite graph and  $2|E(T)|$  divides  $|E(G)|$ , then  $G$  admits a  $T$ -decomposition.*

We prove Conjecture 7.2 restricted to the case  $T$  is a path of even length. The following result states the equivalence of Conjecture 7.2 and Conjecture 1.2.

**Theorem 7.3.** *Let  $T$  be a tree with  $\ell$  edges,  $\ell \geq 3$ . The following two statements are equivalent.*

- (i) *There exists a positive integer  $k_T''$  such that, if  $G$  is a  $k_T''$ -edge-connected bipartite graph and  $2|E(T)|$  divides  $|E(G)|$ , then  $G$  admits a  $T$ -decomposition.*
- (ii) *There exists a positive integer  $k_T'$  such that, if  $G$  is a  $k_T'$ -edge-connected bipartite graph and  $|E(T)|$  divides  $|E(G)|$ , then  $G$  admits a  $T$ -decomposition.*

Furthermore,  $k_T' \leq 2(k_T'' + \ell)$ .

*Proof.* It suffices to prove that statement (i) implies statement (ii). Suppose statement (i) is true. Let  $G$  be an  $2(k_T'' + \ell)$ -edge-connected graph such that  $|E(G)|$  is divisible by  $\ell$ . If  $|E(G)|$  is divisible by  $2\ell$ , then  $G$  satisfies the conditions of statement (i), and therefore admits a  $T$ -decomposition. Thus, we may assume that  $|E(G)| = 2r\ell + \ell$  for some positive integer  $r$ . Clearly,  $G$  contains a copy  $T'$  of  $T$ . By Theorem 2.4,  $G$  contains at least  $k_T'' + \ell$  edge-disjoint spanning trees. Since  $T'$  has  $\ell$  edges,  $T'$  intercepts at most  $\ell$  of these spanning trees, and the graph  $G' = G - T'$  contains at least  $k_T''$  of these spanning trees. Thus,  $G'$  is  $k_T''$ -edge-connected. Note that  $|E(G')| = |E(G)| - \ell = 2r\ell$ . By statement (i),  $G'$  admits a decomposition  $\mathcal{D}'$  into copies of  $T$ . Thus,  $\mathcal{D} = \mathcal{D}' + T'$  is a decomposition of  $G$  into copies of  $T$ .  $\square$



**Theorem 7.4.** *Let  $k$  be a positive integer and let  $r = \max\{32(2k+1), (2k+2)(2k+4)\}$ . If  $G$  is a bipartite  $2(12k+2r+10)$ -edge-connected graph such that  $|E(G)|$  is divisible by  $4k+4$ , then  $G$  admits a decomposition into  $(2k+2)$ -paths.*

*Proof.* Let  $k$  be a positive integer and let  $G$  be a bipartite graph such that  $|E(G)|$  is divisible by  $4k+4 = 2((2k+1)+2)$ . Let  $r = \max\{32(2k+1), (2k+2)(2k+4)\}$ . Suppose that  $G$  is a  $2(12k+2r+10)$ -edge-connected graph, i.e,  $G$  is  $2(6(2k+1)+2r+4)$ -edge-connected.

By Lemma 4.9 (applied with  $2k+1$ ),  $G$  is strongly  $(2, 2(2k+2))$ -bifactorable. Let  $\mathbb{F}$  be a strong  $(2, 2(2k+2))$ -bifactorization of  $G$ . By Theorem 6.5,  $G$  admits an  $\mathbb{F}$ -balanced  $(2k+2)$ -path decomposition.  $\square$

**Corollary 7.5.** *Let  $k$  be a positive integer and let  $r = \max\{32(2k+1), (2k+2)(2k+4)\}$ . If  $G$  is a bipartite  $2(26k+4r+22)$ -edge-connected graph such that  $|E(G)|$  is divisible by  $2k+2$ , then  $G$  admits a decomposition into  $(2k+2)$ -paths.*

*Proof.* The result follows directly from Theorem 7.3.  $\square$

## 8. CONCLUDING REMARKS

This paper benefited greatly from Thomassen's results on decomposition of highly edge-connected graphs. We hope that the connection of this work to these results of Thomassen is clear to a reader familiarized with them. We also would like to mention that Merker's result [17] contributes to the literature with an alternative to the factorizations results presented in this paper. If one can generalize the Disentangling Lemma to deal with general trees, Merker's result can be applied to solve Conjecture 1.2.

While writing this paper we learned that Bensmail, Harutyunyan, Le, and Thomassé [4] obtained a result similar to the one presented here using a different approach.

The Disentangling Lemma has shown to be a powerful technique to deal with path decompositions. We were able to use a version of it to decompose regular graphs with prescribed girths into paths of fixed length [7].

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